

EXPLICIT FORMULAS FOR P -ADIC INTEGRALS: APPROACH TO P -ADIC DISTRIBUTIONS AND SOME FAMILIES OF SPECIAL NUMBERS AND POLYNOMIALS

YILMAZ SIMSEK

ABSTRACT. The main objective of this article is to give and classify new formulas of p -adic integrals and blend these formulas with previously well known formulas. Therefore, this article gives briefly the formulas of p -adic integrals which were found previously, as well as applying the integral equations to the generating functions and other special functions, giving proofs of the new interesting and novel formulas. The p -adic integral formulas provided in this article contain several important well-known families of special numbers and special polynomials such as the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums. In addition, we defined two new sequences containing the Bernoulli numbers of the first kind and the Euler numbers of the first kind. These two sequences include central factorial numbers, Bernoulli numbers of the first kind and the Euler numbers of the first kind. Some computation formulas and identities for these sequences are given. Finally, we give further remarks, observations and comments related to content of this paper.

Keywords: p -adic q -integrals, Volkenborn integral, Generating function, Special functions, Bernoulli numbers and polynomials of the first kind, Euler numbers and polynomials of the first kind, Stirling numbers, Lah numbers, Peters numbers and polynomials, Central factorial numbers, Daehee numbers and polynomials, Changhee numbers and polynomials, Harmonic numbers, Fubini numbers, Combinatorial numbers and sums.

MSC(2010): 11S80, 11B68, 05A15, 05A19, 11M35, 30C15, 26C05, 12D10, 33C45.

1. Introduction

Just before the turn of the 20th century, around the end of the 19th century, Kurt Hensel (1861–1941) constructed a new special number family, now called p -adic numbers. Although p -adic numbers have been known for nearly a hundred years, it is well-known that these numbers have recently

Date: Received: 2 September 2019, Accepted: 9 November 2019.

been applied in the fields of physics, mathematics and other engineering, for example, coding theory and Diophantine equations. The mystery and development of p -adic numbers in science is still growing rapidly. In this way, many scientists continue to be the source of inspiration. Consequently, these numbers are used in several mathematical fields such as Number Theory, Algebraic Geometry, Algebraic Topology, Mathematical Analysis, Mathematical Physics, String Theory, Field Theory, Stochastic Differential Equations on real Banach Spaces and Manifolds, and other parts of the natural sciences. In addition to this rapid development of p -adic numbers, very important theories have been built in p -adic analysis and their applications. These include p -adic distributions and p -adic measure, p -adic integrals, p -adic L -function, and other generalized functions. In order to solve mathematical and physical problems, p -adic numbers are used. A connection between p -adic Analysis and Quantum Groups with Noncommutative Geometry, q -deformation of ordinary analysis has recently given (*cf.* [4], [52]-[81], [74]-[73], [89], [98], [100]-[130]). These major advances, especially the p -adic integral and applications of p -adic analysis, were greatly influenced. The p -adic integrals and their applications are of great importance to find solutions to special (differential) equations, solutions to real world problems in both mathematics, physics and engineering. In recent years, many books, scientific articles, theses and reports have been published on p -adic integrals and their applications. The p -adic integral and generating functions have been used in mathematics, in mathematical physics and in others sciences. From another perspective, the p -adic integral and p -adic numbers are also intensively used in the theory of ultrametric calculus, the p -adic quantum mechanics and the p -adic mechanics (*cf.* [4], [52]-[81], [74]-[73], [89], [98], [100]-[130]).

This paper aims to give comprehensive study of formulas and identities on theory of not only the Volkenborn integral, but also the fermionic p -adic integral and also (p -adic) distributions. Therefore, in addition to the well-known formulas including the Volkenborn integral and the fermionic p -adic integral, which we have found so far in various sources, we will give new and comprehensive formulas that we have found in this study. We hope and believe that these formulas have the enough quality and depth to be used in the fields of p -adic numbers and p -adic analysis we have mentioned above. The content of this paper is including a brief summary of generating functions for special numbers and polynomials, p -adic distribution, the Volkenborn integral and the fermionic p -adic integral. Our new and novel p -adic integral formulas involving the Bernoulli numbers and polynomials of the first kind, the Euler numbers and polynomials of the first kind, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

The following definitions relations and notations are used all sections of this article:

Let \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers, respectively. Additionally, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $n, k \in \mathbb{N}_0$, the binomial coefficient $\binom{n}{k}$ is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n$ or $k < 0$, then we assume that

$$\binom{n}{k} = 0$$

(cf. [2]-[132]).

1.1. Rising and Falling Factorials

Let $x \in \mathbb{R}$. The rising factorial and the falling factorial are defined as follows, respectively:

$$(1.1) \quad x^{(n)} = \begin{cases} x(x+1)(x+2)\dots(x+n-1) & \text{if } n \in \mathbb{N} \\ 1 & \text{if } n = 0 \end{cases},$$

and

$$(1.2) \quad x_{(n)} = \begin{cases} x(x-1)(x-2)\dots(x-n+1) & \text{if } n \in \mathbb{N} \\ 1 & \text{if } n = 0 \end{cases}$$

For $n \in \mathbb{N}_0$, we have

$$(1.3) \quad (-1)^n (-x)_{(n)} = (x+n-1)_{(n)} = x^{(n)}$$

(cf. [2]-[132]).

In order to define the central factorials of degree n , we need the following another type of falling factorial:

$$(1.4) \quad x^{[n]} = x(x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2)\dots(x - \frac{n}{2} + 1)$$

and

$$x^{[0]} = 1$$

where $n \in \mathbb{N}$, $x \in \mathbb{R}$ (cf. [11], [24], [15], [16], [18], [123]).

Using (1.2), we have the following well-known identities and relations:

$$(1.5) \quad xx_{(n)} = x_{(n+1)} + nx_{(n)},$$

$$(1.6) \quad x_{(n+1)} = x \sum_{k=0}^n (-1)^{n-k} n_{(n-k)} x_{(k)},$$

(cf. [99, p. 58]). Additionally, we have

$$(1.7) \quad (x+1)_{(n+1)} = xx_{(n)} + x_{(n)},$$

and

$$(1.8) \quad x_{(m)}x_{(n)} = \sum_{k=0}^m \binom{m}{k} \binom{n}{k} k! x_{(m+n-k)},$$

(cf. [135]).

It is well-known that the coefficients of $x_{(n+n-k)}$ are called connection coefficients and they have a combinatorial interpretation as the number of ways to identify k elements each from a set of size m and a set of size n (cf. [135]).

The well-known Chu-Vandermonde identity is defined as follows:

$$(1.9) \quad \sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}.$$

By (1.9), we have

$$(1.10) \quad \sum_{k=0}^n \binom{x}{k} \binom{m}{n-k} = \binom{x+m}{n},$$

$$\Delta \binom{x}{n} = \binom{x}{n-1}$$

and

$$\Delta \binom{x}{n} = \binom{x+1}{n} - \binom{x}{n}.$$

Therefore

$$(1.11) \quad \binom{x+1}{n} = \binom{x}{n} + \binom{x}{n-1}$$

(cf. [45, p. 69, Eq-(7)]). By

$$(1.12) \quad \Delta x_{(n)} = (x+1)_{(n)} - x_{(n)},$$

we have

$$(1.13) \quad (x+1)_{(n)} = x_{(n)} + nx_{(n-1)}$$

(cf. [99, p. 58]).

Thanks to the works [33] and [34] of Gould, it is also known that the following formulas hold true:

$$(1.14) \quad x \binom{x-2}{n-1} = \sum_{k=1}^n (-1)^{k-n} \binom{x}{k} k$$

(cf. [33, Vol. 3, Eq-(4.20)]),

$$(1.15) \quad \binom{n-x}{n} = \sum_{k=0}^n (-1)^k \binom{x}{k}$$

(cf. [33, Vol. 3, Eq-(4.19)]),

$$(1.16) \quad \binom{mx}{n} = \sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{mk - mj}{n}$$

(cf. [34, Eq-(2.65)]),

$$(1.17) \quad \binom{x}{n}^r = \sum_{k=0}^{nr} \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r$$

(cf. [34, Eq-(2.66)]).

$$(1.18) \quad x \binom{x-2}{n-1} + x(x-1) \binom{n-3}{n-2} = \sum_{k=0}^n (-1)^k \binom{x}{k} k^2,$$

where $n \in \mathbb{N}$ with $n > 1$ (cf. [34, Eq-(2.15)]),

$$(1.19) \quad \binom{x+n}{n} = \sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j+n}{n}$$

and

$$(1.20) \quad \binom{x+n}{n} = \sum_{k=0}^n x^k \sum_{j=0}^n \binom{n}{j} \frac{S_1(j, k)}{j!}$$

(cf. [34, Eq-(2.64) and Eq-(6.17)]),

$$(1.21) \quad \binom{x+n+\frac{1}{2}}{n} = (2n+1) \binom{2n}{n} \sum_{k=0}^n \binom{n}{k} \binom{x}{k} \frac{2^{2k-2n}}{(2k+1) \binom{2k}{k}}$$

(cf. [33, Vol. 3, Eq-(6.26)]).

1.2. Generating Functions for Special Numbers and Polynomials

Here, we give some well-known generating functions which are for special numbers and polynomials.

The Apostol-Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$ are defined by means of the following generating function:

$$(1.22) \quad F_A(t, x; \lambda) = \frac{t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{t^n}{n!},$$

(cf. [7], [123]).

Substituting $x = 0$ into (1.22), we have

$$\lambda \mathcal{B}_1(1; \lambda) = 1 + \mathcal{B}_1(\lambda)$$

and for $n \geq 2$,

$$\lambda \mathcal{B}_n(1; \lambda) = \mathcal{B}_n(\lambda),$$

(cf. [7]).

By using (1.22), we have

$$(1.23) \quad \mathcal{B}_n(x; \lambda) = \sum_{j=0}^n \binom{n}{j} x^{n-j} \mathcal{B}_j(\lambda)$$

(cf. [7]). By using (1.23), first few values of the Apostol-Bernoulli polynomials are given as follows:

$$\begin{aligned} \mathcal{B}_0(x; \lambda) &= 0, \\ \mathcal{B}_1(x; \lambda) &= \frac{1}{\lambda - 1}, \\ \mathcal{B}_2(x; \lambda) &= \frac{1}{\lambda - 1}x - \frac{2\lambda}{(\lambda - 1)^2}, \\ \mathcal{B}_3(x; \lambda) &= \frac{3}{\lambda - 1}x^2 - \frac{6\lambda}{(\lambda - 1)^2}x + \frac{3\lambda(\lambda + 1)}{(\lambda - 1)^3}, \\ \mathcal{B}_4(x; \lambda) &= \frac{4}{\lambda - 1}x^3 - \frac{12\lambda}{(\lambda - 1)^2}x^2 + \frac{12\lambda(\lambda + 1)}{(\lambda - 1)^3}x - \frac{4\lambda(\lambda^2 + 4\lambda + 1)}{(\lambda - 1)^4}, \\ \mathcal{B}_5(x; \lambda) &= \frac{5}{\lambda - 1}x^4 - \frac{20\lambda}{(\lambda - 1)^2}x^3 + \frac{30\lambda(\lambda + 1)}{(\lambda - 1)^3}x^2 - \frac{20\lambda(\lambda^2 + 4\lambda + 1)}{(\lambda - 1)^4}x \\ &\quad + \frac{5\lambda(\lambda^3 + 11\lambda^2 + 11\lambda + 1)}{(\lambda - 1)^5}, \dots \end{aligned}$$

Substituting $x = 1$ into (1.22), we have the following Apostol-Bernoulli numbers:

$$\mathcal{B}_n(1, \lambda) = \sum_{j=0}^n \binom{n}{j} \mathcal{B}_j(\lambda),$$

where

$$\mathcal{B}_n(\lambda) = \mathcal{B}_n(0, \lambda)$$

and

$$\mathcal{B}_0(\lambda) = 0.$$

Since

$$(1.24) \quad \begin{aligned} \mathcal{B}_1(\lambda) &= \frac{1}{\lambda - 1}, \\ \mathcal{B}_m(\lambda) &= \frac{\lambda}{1 - \lambda} \sum_{j=0}^{m-1} \binom{m}{j} \mathcal{B}_j(\lambda), \end{aligned}$$

we have the following few values of the Apostol-Bernoulli numbers:

$$\begin{aligned}\mathcal{B}_2(\lambda) &= \frac{-2\lambda}{(\lambda-1)^2}, \\ \mathcal{B}_3(\lambda) &= \frac{3\lambda(\lambda+1)}{(\lambda-1)^3}, \\ \mathcal{B}_4(\lambda) &= \frac{-4\lambda(\lambda^2+4\lambda+1)}{(\lambda-1)^4}, \\ \mathcal{B}_5(\lambda) &= \frac{5\lambda(\lambda^3+11\lambda^2+11\lambda+1)}{(t-1)^5}, \\ \mathcal{B}_6(\lambda) &= \frac{-6\lambda(\lambda^4+26\lambda^3+66\lambda^2+26\lambda+1)}{(t-1)^6}, \\ \mathcal{B}_7(\lambda) &= \frac{7\lambda(\lambda^5+57\lambda^4+302\lambda^3+302\lambda^2+57\lambda+1)}{(\lambda-1)^7}, \dots\end{aligned}$$

(cf. [7], for detail, see also [59], [39], [83], [122], [124]). When $\lambda = 1$ in (1.22), we have the Bernoulli polynomials of the first kind

$$B_n(x) = \mathcal{B}_n(x; 1).$$

Hence, few values of the Bernoulli polynomials of the first kind are given as follows:

$$\begin{aligned}B_0(x) &= 1, \\ B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x + \frac{1}{6}, \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\ B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \\ B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\ B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}, \dots\end{aligned}$$

Since $B_n = B_n(0)$ denotes the Bernoulli numbers of the first kind, few of these numbers are given as follows:

$$\begin{aligned}B_0 &= 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \\ B_6 &= \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6}, \\ B_{16} &= -\frac{3617}{510}, B_{18} = \frac{43867}{798}, B_{20} = -\frac{174611}{330}, \dots\end{aligned}$$

with $B_{2n+1} = 0$ for $n \geq 2$ (cf. OEIS A000367, OEIS A002445; and also see [8]-[132]; see also the references cited in each of these works).

The λ -Bernoulli polynomials (Apostol-type Bernoulli) polynomials $\mathfrak{B}_n(x; \lambda)$ are defined by means of the following generating function:

$$(1.25) \quad F_B(t, x; \lambda) = \frac{\log \lambda + t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda) \frac{t^n}{n!},$$

(cf. [72]; see also [41], [124], [112]). For $n > 1$, combining (1.25) with (1.22), we have the following well-known identity:

$$\mathfrak{B}_{n-1}(x; \lambda) = \frac{\log \lambda}{n} \mathcal{B}_n(x; \lambda) + \mathcal{B}_{n-1}(x; \lambda).$$

The Apostol-Euler polynomials of the first kind $\mathcal{E}_n(x, \lambda)$ are defined by means of the following generating function:

$$(1.26) \quad F_{P1}(t, x; k, \lambda) = \frac{2}{\lambda e^t + 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_n(x, \lambda) \frac{t^n}{n!},$$

and by using (1.26), we have

$$(1.27) \quad \mathcal{E}_n(x; \lambda) = \sum_{j=0}^n \binom{n}{j} x^{n-j} \mathcal{E}_j(\lambda)$$

(cf. [8]-[125]).

By combining (1.26) with (1.22), we have the following well-known relation:

$$(1.28) \quad \mathcal{E}_n(x; \lambda) = -\frac{2}{n+1} \mathcal{B}_{n+1}(x; -\lambda)$$

(cf. [123]).

Substituting $\lambda = 1$ into (1.26), we have the Euler polynomials of the first kind; that is

$$E_n(x) = \mathcal{E}_n(x; 1).$$

In the light of this thought and also with the help of the equation (1.27), few values of the Euler polynomials of the first kind are given as follows:

$$\begin{aligned} E_0(x) &= 1, \\ E_1(x) &= x - \frac{1}{2}, \\ E_2(x) &= x^2 - x, \\ E_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \\ E_4(x) &= x^4 - 2x^3 + x, \\ E_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}, \dots \end{aligned}$$

Substituting $x = 0$ into (1.26), we have the Apostol-Euler numbers of the first kind:

$$\mathcal{E}_n(\lambda) = \mathcal{E}_n(0, \lambda).$$

Hence

$$(1.29) \quad \begin{aligned} \mathcal{E}_0(\lambda) &= \frac{2}{\lambda + 1}, \\ \mathcal{E}_m(\lambda) &= -\frac{\lambda}{1 + \lambda} \sum_{j=0}^{m-1} \binom{m}{j} \mathcal{E}_j(\lambda). \end{aligned}$$

Using (1.29), we have

$$\begin{aligned} \mathcal{E}_1(\lambda) &= -\frac{2\lambda}{(\lambda + 1)^2}, \\ \mathcal{E}_2(\lambda) &= \frac{2\lambda(\lambda - 1)}{(\lambda + 1)^3}, \\ \mathcal{E}_3(\lambda) &= -\frac{2\lambda(\lambda^2 - 4\lambda + 1)}{(\lambda + 1)^4}, \dots \end{aligned}$$

Setting $\lambda = 1$ into (1.26), we have the Euler numbers of the first kind:

$$E_n = \mathcal{E}_n(1) = E_n(0).$$

Hence few of values of the Euler numbers of the first kind are given as follows:

$$\begin{aligned} E_0 &= 1, E_1 = -\frac{1}{2}, E_2 = 0, E_3 = \frac{1}{4}, \\ E_5 &= -\frac{1}{2}, E_7 = \frac{17}{8}, E_9 = -\frac{31}{2}, \dots \end{aligned}$$

with $E_{2n} = 0$ for $n \geq 1$ (cf. [8]-[125]; see also the references cited in each of these earlier works).

Let u be a complex numbers with $u \neq 1$. The Frobenius-Euler numbers $H_n(u)$ are defined by means of the following generating function:

$$(1.30) \quad F_f(t, u) = \frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}.$$

By using (1.30), we have

$$H_n(u) = \begin{cases} 1 & \text{for } n = 0 \\ \frac{1}{u} \sum_{j=0}^n \binom{n}{j} H_j(u) & \text{for } n > 0. \end{cases}$$

By using the above formula, some values of the numbers $H_n(u)$ are given as follows:

$$\begin{aligned} H_1(u) &= \frac{1}{u-1}, \\ H_2(u) &= \frac{u+1}{(u-1)^2}, \\ H_3(u) &= \frac{u^2+4u+1}{(u-1)^3}, \\ H_4(u) &= \frac{u^3+11u^2+11u+1}{(u-1)^4}, \dots \end{aligned}$$

Substituting $u = -1$ into (1.30), we have

$$E_n = H_n(-1)$$

(cf. [48], [72, Theorem 1, p. 439], [124]; see also the references cited in each of these earlier works).

The Euler numbers of the second kind E_n^* are given by means of the following generating function:

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}.$$

Since

$$E_n^* = 2^n E_n \left(\frac{1}{2} \right),$$

and using the definition of the Euler polynomials the first kind $E_n(x)$, it is easy to give few values of the Euler numbers of the second kind E_n^* given as follows:

$$\begin{aligned} E_0^* &= 1, E_2^* = -1, E_4^* = 5, E_6^* = -61, E_8^* = 1385, E_{10}^* = -50521, \\ E_{12}^* &= 2702765, E_{14}^* = -199360981, E_{16}^* = 19391512145, \dots \end{aligned}$$

with $E_{2n+1}^* = 0$ for $n \geq 0$ (cf. [8]-[125]; see also the references cited in each of these earlier works).

The Bernstein basis functions $B_k^n(x)$ are defined by means of the following generating functions:

$$\frac{(tx)^k}{k!} e^{(1-x)t} = \sum_{n=0}^{\infty} B_k^n(x)$$

where

$$(1.31) \quad B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

$n, k \in \mathbb{N}_0$ with $0 \leq k \leq n$ (cf. [1], [66], [82], [119], [105]).

Note that there is one generating function for each value of k .

The Stirling numbers of the first kind $S_1(n, k)$ are defined by means of the following generating function:

$$(1.32) \quad F_{S_1}(t, k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n, k) \frac{t^n}{n!}.$$

Some basic properties of these numbers are given as follows:

$$S_1(0, 0) = 1$$

and

$$S_1(0, k) = 0$$

if $k > 0$. Also

$$S_1(n, 0) = 0$$

if $n > 0$ and

$$S_1(n, k) = 0$$

if $k > n$. A recurrence relation for these numbers is given by

$$(1.33) \quad S_1(n+1, k) = -nS_1(n, k) + S_1(n, k-1)$$

(cf. [15], [8], [13], [99], [104], [107]; and see also the references cited in each of these earlier works).

By using (1.33), few values of the Stirling numbers of the first kind $S_1(n, k)$ are given by the following table:

$n \backslash k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	-1	1	0	0	0
3	0	2	-3	1	0	0
4	0	-6	11	-6	1	0
5	0	24	-50	35	-10	1

Another generating function for the Stirling numbers of the first kind is falling factorial function which is given as follows:

$$(1.34) \quad x_{(n)} = \sum_{k=0}^n S_1(n, k) x^k$$

(cf. [15], [16], [18], [123]).

Some well-known identities for the equation (1.34) are given as follows:

Multiplying both sides of the equation (1.34) by x^m , we have

$$(1.35) \quad x^m x_{(n)} = \sum_{k=0}^n S_1(n, k) x^{m+k}.$$

By combining (1.8) with (1.34), we have

$$(1.36) \quad x_{(m)} x_{(n)} = \sum_{k=0}^m \binom{m}{k} \binom{n}{k} k! \sum_{l=0}^{m+n-k} S_1(m+n-k, l) x^l.$$

By using (1.34), we have

$$(1.37) \quad x_{(m)}x_{(n)} = \sum_{j=0}^n \sum_{l=0}^m S_1(n, k)S_1(m, l)x^{j+l}.$$

By combining (1.5) with (1.34), we have

$$(1.38) \quad xx_{(n)} = \sum_{k=0}^n (S_1(n+1, k) + nS_1(n, k))x^k + x^{n+1}.$$

By combining the above equation with (1.33), and using $S_1(n, k) = 0$ if $k < 0$, we have

$$(1.39) \quad xx_{(n)} = \sum_{k=1}^n S_1(n, k-1)x^k + x^{n+1}.$$

The unsigned Stirling numbers of the first kind are defined by

$$C(n, k) = |S_1(n, k)| = \left[\begin{matrix} n \\ k \end{matrix} \right]$$

(cf. [15], [16], [18], [123]). The numbers $C(n, k)$ are also defined as follows:

$$(1.40) \quad x^{(n)} = \sum_{k=0}^n C(n, k)x^k$$

(cf. [14]).

The Bernoulli polynomials of the second kind $b_n(x)$ are defined by means of the following generating function:

$$(1.41) \quad F_{b2}(t, x) = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}$$

(cf. [99, pp. 113-117]; see also the references cited in each of these earlier works).

The Bernoulli numbers of the second kind $b_n(0)$ are defined by means of the following generating function:

$$(1.42) \quad F_{b2}(t) = \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n(0) \frac{t^n}{n!}.$$

The Bernoulli polynomials of the second kind are defined by

$$b_n(x) = \int_x^{x+1} u_{(n)} du.$$

Hence few of values of the polynomials $b_n(x)$ are given as follows:

$$\begin{aligned} b_0(x) &= 1, \\ b_1(x) &= x + \frac{1}{2}, \\ b_2(x) &= x^2 - \frac{1}{6}, \\ b_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \\ b_4(x) &= x^4 - 4x^3 + 4x^2 - \frac{19}{30}, \dots \end{aligned}$$

Substituting $x = 0$ into the above equation, one has

$$(1.43) \quad b_n(0) = \int_0^1 u_{(n)} du.$$

The numbers $b_n(0)$ are also so-called Cauchy numbers (i.e. Bernoulli numbers of the second kind). Hence few of values of the numbers $b_n(0)$ are given as follows:

$$\begin{aligned} b_0(0) &= 1, b_1(0) = \frac{1}{2}, b_2(0) = -\frac{1}{6}, b_3(0) = \frac{1}{4}, \\ b_4(0) &= -\frac{19}{30}, b_5(0) = \frac{9}{4}, b_6(0) = -\frac{863}{84}, b_7(0) = \frac{1375}{54}, \dots \end{aligned}$$

(cf. [99, p. 116], [53], [84], [110]; see also the references cited in each of these earlier works).

The λ -array polynomials $S_k^n(x; \lambda)$ are defined by means of the following generating function:

$$(1.44) \quad g_A(t, x, k; \lambda) = \frac{(\lambda e^t - 1)^k}{k!} e^{tx} = \sum_{n=0}^{\infty} S_k^n(x; \lambda) \frac{t^n}{n!},$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (cf. [104], for detail, see also [8], [13], [107]).

By (1.44), we have

$$(1.45) \quad S_k^n(x; \lambda) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \lambda^j (j+x)^n$$

(cf. [104]). Substituting $x = 0$ into (1.44), we have the λ -Stirling numbers $S_2(n, k; \lambda)$, which are defined by the following generating function:

$$(1.46) \quad F_S(t, k; \lambda) = \frac{(\lambda e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k; \lambda) \frac{t^n}{n!},$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (cf. [83], [122], see also [104]).

Substituting $\lambda = 1$ into (1.46), then we get the Stirling numbers of the second kind, which is the number of partitions of a set of n elements into k

nonempty subsets, as follows:

$$(1.47) \quad S_2(n, k) = S_2(n, k; 1) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \lambda^j j^n.$$

The Stirling numbers of the second kind are also given by the following generating function including falling factorial:

$$(1.48) \quad x^n = \sum_{k=0}^n S_2(n, k) x_{(k)},$$

(cf. [7]-[125]; see also the references cited in each of these earlier works).

By using (1.47), few values of the Stirling numbers of the second kind $S_2(n, k)$ are given by the following table:

$n \setminus k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

The Schlomilch formula, associated with the Stirling numbers of the first and the second kind, is given by

$$S_1(n, k) = \sum_{j=0}^{n-k} (-1)^j \binom{n+j-1}{k-1} \binom{2n-k}{n-k-j} S_2(n-k+j, j)$$

(cf. [14, p. 115], [15, p. 290, Eq-(8.21)]).

In [85], Osgood and Wu gave the following identity:

$$(1.49) \quad (xy)_{(k)} = \sum_{l,m=1}^k C_{l,m}^{(k)} x_{(l)} x_{(m)}$$

where

$$C_{l,m}^{(k)} = \sum_{j=1}^k (-1)^{k-j} S_1(k, j) S_2(j, l) S_2(j, m)$$

$$C_{l,m}^{(k)} = C_{m,l}^{(k)}, C_{1,1}^{(1)} = 1, C_{1,1}^{(2)} = 0, C_{1,2}^{(3)} = 0 = C_{2,1}^{(3)}.$$

By using (1.48), we have

$$(1.50) \quad (xy)_{(k)} = \sum_{m=0}^k S_1(k, m) x^m y^m.$$

The Lah numbers are defined by means of the following generating function:

$$(1.51) \quad F_L(t, k) = \frac{1}{k!} \left(\frac{t}{1-t} \right)^k = \sum_{n=k}^{\infty} L(n, k) \frac{t^n}{n!}$$

(cf. [14], [97, p. 44], [9], [92], [15], [18], [92], [137], and the references cited therein).

Using (1.51), we have

$$(1.52) \quad L(n, k) = (-1)^n \frac{n!}{k!} \binom{n-1}{k-1}.$$

The unsigned Lah numbers are defined by

$$(1.53) \quad |L(n, k)| = \frac{n!}{k!} \binom{n-1}{k-1},$$

where $n, k \in \mathbb{N}$ with $1 \leq k \leq n$.

By using (1.53), few values of the unsigned Lah numbers $|L(n, k)|$ are given by the following table:

$n \backslash k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	2	1	0	0	0
3	0	6	6	1	0	0
4	0	24	36	12	1	0
5	0	120	240	120	20	1

By the help of the following the initial conditions

$$L(n, 0) = \delta_{n,0}$$

and

$$L(0, k) = \delta_{0,k},$$

for all $k, n \in \mathbb{N}$, we have recurrence relations for the Lah numbers given as follows:

$$L(n+1, k) = -(n+k)L(n, k) - L(n, k-1)$$

and

$$L(n, k) = \sum_{j=0}^n (-1)^j S_1(n, j) S_2(j, k)$$

(cf. [30], [97, p. 44], [92]).

Using (1.3), we have another definition of the Lah numbers including the falling factorial and the rising factorial:

$$(1.54) \quad (-x)_{(n)} = \sum_{k=1}^n L(n, k) x_{(k)}$$

so that

$$x_{(n)} = \sum_{k=1}^n L(n, k) (-x)_{(k)}$$

and

$$(1.55) \quad x^{(n)} = \sum_{k=1}^n |L(n, k)| x_{(k)}$$

where $n \in \mathbb{N}_0$ (cf. [14], [15], [18], [47], [30], [92], [97], [137]).

The equation (1.4) classification enables us to give the following central factorials of degree n , $t(n, k)$ and $T(n, k)$ of the first and the second kind, respectively:

$$(1.56) \quad x^{[n]} = \sum_{k=0}^n t(n, k)x^k$$

and

$$(1.57) \quad x^n = \sum_{k=0}^n T(n, k)x^{[k]}$$

with

$$t(n, 0) = T(n, 0) = \delta_{0n}$$

where δ_{mn} denotes the Kronecker delta and $n \in \mathbb{N}_0$ (cf. [24]).

Observe that

$$D^j \{x^{[n]}\} = j! \sum_{k=j}^n \binom{k}{j} t(n, k)x^{k-j},$$

$$\delta^j \{x^n\} = j! \sum_{k=j}^n \binom{k}{j} T(n, k)x^{[k-j]},$$

$$\delta^j \{x^n\} |_{x=0} = j!T(n, j),$$

and

$$D^j \{x^{[n]}\} |_{x=0} = j!t(n, j)$$

where

$$\delta \{f(x)\} = f\left(x + \frac{1}{2}\right) - f\left(x - \frac{1}{2}\right)$$

and

$$D = \frac{d}{dx}$$

(cf. [24, Eq. (2.7), Eq. (2.9)], [64], [107]).

Applying the Cauchy's integral theorem to the function $(\sinh(\frac{z}{2}))^m$, we have the well-known integral representations for the numbers $T(n, k)$ and $t(n, k)$, respectively, given as follows:

$$T(n, k) = \frac{k!}{m!2\pi i} \int_{|w|=r} \left(2 \sinh\left(\frac{z}{2}\right)\right)^m \frac{dz}{z^{k+1}}$$

and

$$t(n, k) = \frac{k!}{m!2\pi i} \int_{|w|=r} \left(2 \operatorname{area} \sinh\left(\frac{z}{2}\right)\right)^m \frac{dz}{z^{k+1}}$$

(cf. [24, Proposition 4.2.2]).

The following tables give us the upper part of the matrices of central factorial numbers of the first and second kind, respectively (cf. [16, p. 13]; see also OEIS A036969):

$$(1.58) \quad (T(i, j))_{i, j=0}^6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 21 & 14 & 1 & 0 & 0 \\ 0 & 1 & 85 & 147 & 30 & 1 & 0 \\ 0 & 1 & 341 & 1408 & 627 & 55 & 1 \end{bmatrix}$$

and

$$(1.59) \quad (t(i, j))_{i, j=0}^6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & -5 & 1 & 0 & 0 & 0 \\ 0 & -36 & 49 & -14 & 1 & 0 & 0 \\ 0 & 576 & -870 & 273 & -30 & 1 & 0 \\ 0 & -14400 & 21076 & -7645 & 1023 & -55 & 1 \end{bmatrix}.$$

The Daehee numbers of the first kind and the second kind are defined by means of the following generating functions, respectively:

$$(1.60) \quad \frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}$$

and

$$\frac{(1+t)\log(1+t)}{t} = \sum_{n=0}^{\infty} \widehat{D}_n \frac{t^n}{n!},$$

(cf. [97, p. 45], [28], [49]). Using (1.60), we have

$$D_n = (-1)^n \frac{n!}{n+1}$$

(cf. [49], see also [96], [134], [38]).

The Changhee numbers of the first kind and the second kind are defined by means of the following generating functions, respectively:

$$(1.61) \quad \frac{2}{t+1} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!}$$

and

$$\frac{2(1+t)}{t+2} = \sum_{n=0}^{\infty} \widehat{Ch}_n \frac{t^n}{n!},$$

(cf. [52]). Using (1.61), we have

$$Ch_n = (-1)^n \frac{n!}{2^n}$$

and

$$(1.62) \quad Ch_n = \sum_{k=0}^n S_1(n, k) E_k$$

(*cf.* [52], see also [43], [44]).

The Peters polynomials $s_k(x; \lambda, \mu)$ are defined by means of the following generating function:

$$(1.63) \quad \frac{1}{\left(1 + (1+t)^\lambda\right)^\mu} (1+t)^x = \sum_{n=0}^{\infty} s_n(x; \lambda, \mu) \frac{t^n}{n!}$$

which, for $x = 0$, reduces to the Peters numbers $s_n(0; \lambda, \mu) = s_n(\lambda, \mu)$ (*cf.* [99], [45]), [117]).

Substituting $\mu = 1$ into (1.63), we have the Boole polynomials. If we substitute $\lambda = 1$ and $\mu = 1$ into (1.63), then we have the Changhee polynomials (*cf.* [52], [50]). We observe that recently, the Peters polynomials, the Boole polynomials, the Changhee polynomials, the Daehee polynomials and combinatorial numbers and polynomials have been studied by many authors (see, for details, [27]-[29], [47]-[81]; and also see the other recent works cited in this paper).

By using (1.63), we have

$$\sum_{n=0}^{\infty} s_n(x; \lambda, \mu) \frac{t^n}{n!} = \sum_{n=0}^{\infty} s_n(\lambda, \mu) \frac{t^n}{n!} \sum_{n=0}^{\infty} x_{(n)} \frac{t^n}{n!}.$$

Therefore, we have

$$(1.64) \quad s_n(x; \lambda, \mu) = \sum_{v=0}^n \binom{n}{v} x_{(n-v)} s_v(\lambda, \mu)$$

(*cf.* [99], [45], [50], [117]).

In [114], we defined the generating function for the combinatorial numbers $y_1(n, k; \lambda)$ as follows:

$$(1.65) \quad F_{y_1}(t, k; \lambda) = \frac{1}{k!} (\lambda e^t + 1)^k = \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!},$$

where $k \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ and

$$y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} j^n \lambda^j.$$

By (1.65), we have

$$B(n, k) = k! y_1(n, k; 1)$$

(*cf.* [31], [114]).

Theorem 1.1 (cf. [118]). *Let $\mu \in \mathbb{N}$. Then we have*

$$(1.66) \quad x_{(n)} = \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} s_{n-v}(x; \lambda, \mu).$$

Theorem 1.2 (cf. [118]). *Let $\mu \in \mathbb{Z}^+$. Then we have*

$$(1.67) \quad x_{(n)} = \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) s_{n-v}(x; \lambda, \mu).$$

In [118], we constructed the following generating function for combinatorial polynomials $Y_{n,2}(x; \lambda)$, which are member of the family of the Peters polynomials, as follows:

$$(1.68) \quad F_{Y_2}(t, x; \lambda) = \frac{2(1 + \lambda t)^x}{\lambda^2 t + 2(\lambda - 1)} = \sum_{n=0}^{\infty} Y_{n,2}(x; \lambda) \frac{t^n}{n!}$$

in which if we set $x = 0$, then we have combinatorial numbers $Y_{n,2}(\lambda) = Y_{n,2}(0; \lambda)$. By using (1.68), we have

$$Y_{n,2}(\lambda) = \frac{1}{2^{n+1}} Y_n(\lambda)$$

(cf. [115], [118]) so that the numbers $Y_n(\lambda)$ are defined by the author in (cf. [115]).

Substituting $x = 0$, $\lambda = \mu = 1$ and $t = \frac{\theta^2 u}{\theta - 1}$ into (1.63), we have

$$s_n(0; 1, 1) = \frac{(\theta - 1)^{n+1}}{2\theta^{2n}} Y_{n,2}(\theta)$$

(cf. [118]).

Theorem 1.3 (cf. [118]). *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} s_n(x; \lambda, \mu) &= \frac{n}{2} \sum_{j=0}^{n-1} \binom{n-1}{j} \theta^{j+2-n} s_j(\lambda, \mu) Y_{n-1-j,2}(x; \theta) \\ &\quad + (\theta - 1) \sum_{j=0}^n \binom{n}{j} \theta^{j-n} s_j(\lambda, \mu) Y_{n-j,2}(x; \theta). \end{aligned}$$

Theorem 1.4 (cf. [118]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(1.69) \quad Y_{n,2}(x; \lambda) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) x_{(n-j)}.$$

Lemma 1.5 (cf. [118]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(1.70) \quad Y_{n,2}(\lambda) = 2(-1)^n n! \frac{\lambda^{2n}}{(2\lambda - 2)^{n+1}}.$$

Substituting (1.70) into (1.69), we get a explicit formula for the polynomials $Y_{n,2}(x; \lambda)$ by the following theorem:

Theorem 1.6 (cf. [118]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(1.71) \quad Y_{n,2}(x; \lambda) = 2 \sum_{j=0}^n (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j}}{(2\lambda - 2)^{j+1}} x_{(n-j)}.$$

This paper has exactly 10 main sections including introduction. We summarize as follows:

In Section 2, we give some definitions, notations and formulas for distributions and p -adic (q -) integrals on \mathbb{Z}_p .

In Section 3, we give some applications and formulas for the Volkenborn integral.

In Section 4, we give some applications and formulas for the fermionic p -adic integral.

In Section 5, we give many new formulas for the Volkenborn integral including the falling factorials, the raising factorials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

In Section 6, we give many new formulas for the fermionic p -adic integral including the falling factorials, the raising factorials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

In Section 7, by using formulas in Section 6 and in Section 7, we give various identities including the Bernoulli numbers and polynomials of the first kind, the Euler numbers and polynomials of the first kind, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

In Section 8, we give some questions and new sequences with their definitions and properties.

In Section 9, we give conclusion and observations on our results.

In Section 10, we finalize this paper along with an appendix of some tables, which contains numerous p -adic integral formulas, that may constitute very useful source to researchers working in the related areas. The aforementioned tables include both new and previously known formulas for the Volkenborn integral and the fermionic p -adic integral.

2. Distributions and p -adic q -integrals on \mathbb{Z}_p

In this section, we give brief introduction for p -adic distributions and p -adic (q -) integrals. For the fundamental properties of p -adic integrals and p -adic distributions, which are given briefly below, we may refer the references [4–6, 57, 59, 76, 86, 101, 129, 130]; and the references cited therein.

Some notations and definitions for p -adic integrals are given as follows:

Let p be an odd prime number. Let $m \in \mathbb{N}$. Let $\text{ord}_p(m)$ denote the greatest integer k ($k \in \mathbb{N}_0$) such that p^k divides m in \mathbb{Z} . If $m = 0$, then $\text{ord}_p(m) = \infty$. Let $x \in \mathbb{Q}$, the set of rational numbers, with $x = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ with $b \neq 0$. Therefore,

$$\text{ord}_p(x) = \text{ord}_p\left(\frac{a}{b}\right) = \text{ord}_p(a) - \text{ord}_p(b).$$

Let $|\cdot|_p$ is a map on \mathbb{Q} . This map, which is a norm over \mathbb{Q} , is defined by

$$|x|_p = \begin{cases} p^{-\text{ord}_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For instance, $x \in \mathbb{Q}$ with $x = p^y \frac{x_1}{x_2}$ where $y, x_1, x_2 \in \mathbb{Z}$ and x_1 and x_2 are not divisible by p . Hence, $\text{ord}_p(x) = y$ and $|x|_p = p^{-y}$. The set \mathbb{Q}_p equipped with this norm $|x|_p$ is a topological completion of of set \mathbb{Q} . Let \mathbb{C}_p be the field of p -adic completion of algebraic closure of \mathbb{Q}_p . Let \mathbb{Z}_p be topological closure of \mathbb{Z} . Let \mathbb{Z}_p be a set of p -adic integers, which is related to the norm $|x|_p$, given as follows:

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}$$

In order to define p -adic integral, we need the following definitions and formulas.

Let $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$. f is called a uniformly differential function at a point $a \in \mathbb{Z}_p$ if f satisfies the following conditions:

If the difference quotients $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{C}_p$ such that

$$\Phi_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

have a limit $f'(z)$ as $(x, y) \rightarrow (0, 0)$ (x and y remaining distinct). A set of uniformly differential functions is briefly indicated by $f \in UD(\mathbb{Z}_p)$ or $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p)$. The additive cosets of \mathbb{Z}_p are given as follows:

$$p\mathbb{Z}_p = \left\{ x \in \mathbb{Z}_p : |x|_p < 1 \right\}, 1 + p\mathbb{Z}_p, \dots, p-1 + p\mathbb{Z}_p,$$

where $p\mathbb{Z}_p$ is a maximal ideal of \mathbb{Z}_p and for each $j \in \{0, 1, \dots, p^n - 1\}$ we set

$$j + p^n\mathbb{Z}_p = \left\{ x \in \mathbb{Z}_p : |x - j|_p < p^{1-n} \right\}.$$

Thus, we have

$$\mathbb{Z}_p = \cup_{j=0}^{p-1} (j + p\mathbb{Z}_p).$$

By using the above coset, we give the following distributions on \mathbb{Z}_p for p -adic integrals:

Every map μ from the set of intervals contained in X to \mathbb{Q}_p for which

$$\mu(x + p^n \mathbb{Z}_p) = \sum_{j=0}^{p-1} \mu(x + jp^n + p^{n+1} \mathbb{Z}_p)$$

whenever $x + p^n \mathbb{Z}_p \subset X$, exists uniquely to a p -adic distribution on X (cf. [4], [78], [101], [120], [129], [130]).

Some well-known examples for the p -adic distribution are given as follows:

The Haar distribution is defined by

$$(2.1) \quad \mu_{Haar}(x + p^N \mathbb{Z}_p) = \mu_1(x + p^N \mathbb{Z}_p) = \frac{1}{p^N},$$

which denotes by $\mu_1(x + p^N \mathbb{Z}_p) = \mu_1(x)$;

The Dirac distribution is defined by

$$\mu_{Dirac}(x + p^N \mathbb{Z}_p) = \mu_\alpha(X) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{otherwise;} \end{cases}$$

The Mazur distribution is defined by

$$\mu_{Mazur}(x + p^N \mathbb{Z}_p) = \frac{a}{p^N} - \frac{1}{2},$$

where $a \in \mathbb{Q}$ with $0 \leq a \leq p-1$;

The Bernoulli distribution is defined by

$$\mu_{B,k}(x + p^N \mathbb{Z}_p) = p^{N(k-1)} B_k\left(\frac{a}{p^N}\right),$$

(cf. [4], [56], [75], [78], [101], [120], [129], [130]).

Observe that special values of the Bernoulli distribution are related to the Haar distribution and the Mazur distribution; that is

$$\mu_{B,0}(x + p^N \mathbb{Z}_p) = \mu_1(x + p^N \mathbb{Z}_p)$$

and

$$\mu_{B,1}(x + p^N \mathbb{Z}_p) = \mu_{Mazur}(x + p^N \mathbb{Z}_p)$$

(cf. [78, p.35]).

On the other hand, we have the following distribution $\mu_{-1}(x + p^N \mathbb{Z}_p)$ on \mathbb{Z}_p :

$$(2.2) \quad \mu_{-1}(x + p^N \mathbb{Z}_p) = (-1)^x$$

which is denoted by $\mu_{-1}(x + p^N \mathbb{Z}_p) = \mu_{-1}(x)$ (cf. [74], [73], [57], [59], [95]).

The Euler distribution is defined by

$$\mu_{\mathcal{E},k,q}(x + fp^N \mathbb{Z}_p) = (-1)^a \left(fp^N\right)^k \mathcal{E}_k\left(\frac{a}{fp^N}; q^{fp^N}\right),$$

where $N, k, f \in \mathbb{N}$ and f is odd (cf. [95], [88], [87]).

Therefore

$$(2.3) \quad \begin{aligned} \lim_{q \rightarrow 1} \mu_{\mathcal{E},k,q}(x + fp^N \mathbb{Z}_p) &= \mu_{E,k}(x + fp^N \mathbb{Z}_p) \\ &= (-1)^x (fp^N)^k E_k \left(\frac{x}{fp^N} \right), \end{aligned}$$

(cf. [95]).

Since

$$|\mu_{E,k}(x + fp^N \mathbb{Z}_p)|_p \leq 1,$$

$\mu_{E,k}(x + fp^N \mathbb{Z}_p)$ is a measure on \mathbb{X} where

$$\mathbb{X} = \mathbb{X}_f = \varprojlim_{\mathbb{N}} \mathbb{Z}/fp^N \mathbb{Z} \text{ and } \mathbb{X}_1 = \mathbb{Z}_p$$

(cf. [95]; see also [75], [88], [87]).

Substituting $f = 1$ and $k = 0$ into (2.3), we have

$$\mu_{E,0}(x + fp^N \mathbb{Z}_p) = \mu_{-1}(x).$$

In order to define invariant p -adic z -integrals, Kim [58] gave the following distribution on \mathbb{Z}_p :

Let

$$\mu_z(a + p^N \mathbb{Z}_p) = \frac{z^a}{[p^N : z]},$$

where

$$[x : z] = \frac{1 - z^x}{1 - z}.$$

It well-known that $\mu_z(x + p^N \mathbb{Z}_p) = \mu_z(x)$ is extended distribution on \mathbb{Z}_p (cf. [124, p. 244]).

For a compact-open subset \mathbb{X} of \mathbb{Q}_p , a p -adic distribution μ on \mathbb{X} is a \mathbb{Q}_p -linear vector space homomorphism from the \mathbb{Q}_p -vector space of locally constant functions on \mathbb{X} to \mathbb{Q}_p (cf. [101]).

Let \mathbb{K} be a field with a complete valuation and $C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$ be a set of functions which have continuous derivative (see, for detail, [101]).

Kim [57] defined the p -adic q -integral as follows:

Let $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. Then we have

$$(2.4) \quad I_q(f(x)) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x,$$

where

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q}, & q \neq 1 \\ x, & q = 1 \end{cases}$$

and

$$\mu_q(x) = \mu_q(x + p^N \mathbb{Z}_p)$$

which denotes q -distribution on \mathbb{Z}_p and it is defined by

$$\mu_q(x + p^N \mathbb{Z}_p) = \frac{q^x}{[p^N]_q},$$

(cf. [57]).

Observe that

$$\lim_{q \rightarrow 1} \mu_q(x + p^N \mathbb{Z}_p) = \mu_{Haar}(x + p^N \mathbb{Z}_p) = \mu_1(x)$$

and

$$\lim_{q \rightarrow -1} \mu_q(x + p^N \mathbb{Z}_p) = \mu_{-1}(x).$$

Observe that if $q \rightarrow 1$, then (2.4) reduces to the following well-known Volkenborn integral (bosonic integral), which is denoted by $I_1(f(x))$:

$$(2.5) \quad \lim_{q \rightarrow 1} I_q(f(x)) = I_1(f(x)) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),$$

where $\mu_1(x)$ is given by the equation (2.1), that is

$$\mu_1(x) = \frac{1}{p^N}$$

(cf. [4], [76], [101], [129], [130]); see also the references cited in each of these earlier works).

The above integral has many applications not only in mathematics, but also in mathematical physics. By using this integral and its integral equations, various family of generating functions associated with Bernoulli-type numbers and polynomials have been constructed (cf. [7]-[132]).

Over and above, if $q \rightarrow -1$, then (2.4) reduces to the following well-known fermionic p -adic integral, which is denoted by $I_{-1}(f(x))$:

$$(2.6) \quad \begin{aligned} \lim_{q \rightarrow -1} I_q(f(x)) = I_{-1}(f(x)) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) \\ &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} (-1)^x f(x), \end{aligned}$$

where $\mu_{-1}(x)$ is given by the equation (2.2), that is

$$\mu_{-1}(x) = (-1)^x$$

(cf. [59], see also [73]).

By using p -adic fermionic integral and its integral equations, various different generating functions, including Euler-type numbers and polynomials and Genocchi-type numbers and polynomials, have been constructed (cf. [7]-[132]).

We also note that p -adic q -integrals are related to the theory of the generating functions, ultrametric calculus, the quantum groups, cohomology groups, q -deformed oscillator and p -adic models (cf. [76, 129]).

2.1. Some Properties of the Volkenborn Integral

Here, we give some well-known properties of the Volkenborn integral (bosonic p -adic integral).

Let

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K}),$$

where $\binom{x}{n} = \frac{x^{(n)}}{n!}$ the Mahler coefficients. Applying the Volkenborn integral to the function $f(x)$ in terms of the Mahler coefficients $\binom{x}{n}$, we have the following well-known formula:

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} a_n,$$

(cf. [101, p. 168-Proposition 55.3]).

In [101], Schikhof gave the following integral formula for the Volkenborn integral:

$$(2.7) \quad \int_{\mathbb{Z}_p} f(x+n) d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) + \sum_{k=0}^{n-1} f'(k),$$

where

$$f'(x) = \frac{d}{dx} \{f(x)\}.$$

Substituting

$$f(x) = \binom{x}{n}$$

into (2.7), we have

$$\int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1(x) + \frac{d}{dx} \left\{ \binom{x}{n} \right\} \Big|_{x=0},$$

where

$$\begin{aligned} \frac{d}{dx} \left\{ \binom{x}{n} \right\} \Big|_{x=0} &= \left\{ \frac{1}{n!} x^{(n)} \sum_{k=0}^{n-1} \frac{1}{x-k} \right\} \Big|_{x=0} \\ &= (-1)^{n-1} \frac{1}{n}. \end{aligned}$$

Therefore, we have

$$(2.8) \quad \int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = (-1)^n \left(\frac{1}{n+1} - \frac{1}{n} \right).$$

Let $f : \mathbb{Z}_p \rightarrow \mathbb{K}$ be an analytic function and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with $x \in \mathbb{Z}_p$.

The Volkenborn integral of this analytic function is given by

$$\int_{\mathbb{Z}_p} \left(\sum_{n=0}^{\infty} a_n x^n \right) d\mu_1(x) = \sum_{n=0}^{\infty} a_n \int_{\mathbb{Z}_p} x^n d\mu_1(x)$$

and

$$\int_{\mathbb{Z}_p} \left(\sum_{n=0}^{\infty} a_n x^n \right) d\mu_1(x) = \sum_{n=0}^{\infty} a_n B_n,$$

(*cf.* [57], [59], [101]; see also the references cited in each of these earlier works).

Integral equation for the Volkenborn integral is given as follows:

$$(2.9) \quad \int_{\mathbb{Z}_p} E^m [f(x)] d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) + \sum_{j=0}^{m-1} \frac{d}{dx} \{f(x)\} |_{x=j}$$

where

$$E^m [f(x)] = f(x + m)$$

and

$$\frac{d}{dx} \{f(x)\} |_{x=j} = f'(j)$$

(*cf.* [57], [59], [101], [136]; see also the references cited in each of these earlier works).

Using (2.4), the following integral equation was given by Kim [61]:

$$(2.10) \quad q \int_{\mathbb{Z}_p} E [f(x)] d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + \frac{q-1}{\log q} f'(0) + (q-1)f(0)$$

(*cf.* see also [52]-[70]).

As usual, exponential function is defined as follows:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

The above series convergences in region \mathbb{E} which is a subset of field \mathbb{K} with $\text{char}(\mathbb{K}) = 0$ (*cf.* [101, p. 70]). Let k be residue class field of \mathbb{K} . If $\text{char}(k) = p$, then

$$\mathbb{E} = \left\{ x \in \mathbb{K} : |x| < p^{\frac{1}{1-p}} \right\},$$

and if $\text{char}(k) = 0$, then

$$\mathbb{E} = \{x \in \mathbb{K} : |x| < 1\}.$$

Let $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$. Kim [61, Theorem 1] gave the following integral equation:

$$(2.11) \quad \begin{aligned} & q^n \int_{\mathbb{Z}_p} E^n[f(x)] d\mu_q(x) - \int_{\mathbb{Z}_p} f(x) d\mu_q(x) \\ &= \frac{q-1}{\log q} \left(\sum_{j=0}^{n-1} q^j f'(j) + \log q \sum_{j=0}^{n-1} q^j f(j) \right), \end{aligned}$$

where n is a positive integer.

Observe that substituting $n = 1$ into (2.11), we arrive at (2.10).

Theorem 2.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$(2.12) \quad \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1(x) = \frac{(-1)^n}{n+1}.$$

Note that Theorem 2.1 was proved by Schikhof [101].

Substituting $m = 1$ and $f(x) = (1+a)^x$ into (2.9), we have

$$\int_{\mathbb{Z}_p} (1+t)^x d\mu_1(x) = \frac{1}{t} \log(1+t).$$

Therefore,

$$\sum_{n=0}^{\infty} t^n \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1(x) = \frac{1}{t} \log(1+t).$$

Combining the above equation with (2.12), we have the following well-known relation:

$$\log(1+t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

(cf. [101], [136]). We observe that

$$\int_{\mathbb{Z}_p} a^x d\mu_1(x) = \frac{1}{a-1} \log_p(a),$$

where $a \in \mathbb{C}_p^+$ with $a \neq 1$ (cf. [101, p. 170]).

Let $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$. Then we have

$$\int_{\mathbb{Z}_p} f(-x) d\mu_1(x) = \int_{\mathbb{Z}_p} f(1+x) d\mu_1(x)$$

and if $f(-x) = -f(x)$, which is geometrically symmetric about the origin, we have

$$(2.13) \quad \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = -\frac{1}{2} f'(0)$$

(cf. [101, p. 169]). By using the above formulas, we give the following well-known examples:

$$(2.14) \quad \int_{\mathbb{Z}_p} e^{ax} d\mu_1(x) = \frac{a}{e^a - 1} \\ = \sum_{n=0}^{\infty} B_n \frac{a^n}{n!},$$

where $a \in \mathbb{E}$ with $a \neq 0$ (cf. [101, p. 172]). Using Taylor series for e^{ax} in the left-hand side of the equation (2.14), we have the following well-known the Witt's formula for the Bernoulli numbers of the first kind B_n :

$$(2.15) \quad B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x),$$

where $n \in \mathbb{N}_0$ (cf. [101]; see also [57], [59] and the references cited in each of these earlier works). It is well-known that the denominator of the Bernoulli numbers of the first kind B_n is free of square. Consequently, for $n \in \mathbb{N}_0$, $x^n \in C^1(\mathbb{Z}_p \rightarrow \mathbb{Q})$. Then we have

$$|B_n|_p = \left| \int_{\mathbb{Z}_p} x^n d\mu_1(x) \right|_p \leq p$$

(cf. [101, p. 172]). Similarly, we have p -adic representation for the Bernoulli polynomials of the first kind as follows:

$$(2.16) \quad \int_{\mathbb{Z}_p} (z+x)^n d\mu_1(x) = B_n(z)$$

where $n \in \mathbb{N}_0$ (cf. [57], [59], [101]; see also the references cited in each of these earlier works).

Let

$$P_n(x) = \sum_{j=0}^n a_j x^j$$

be a polynomial of degree n ($n \in \mathbb{N}_0$). Substituting $P_n(x)$ into (2.5), we have

$$\int_{\mathbb{Z}_p} P_n(x) d\mu_1(x) = \sum_{j=0}^n a_j \int_{\mathbb{Z}_p} x^j d\mu_1(x).$$

Since $B_{2n+1} = 0$ for $n \in \mathbb{N}$, by combining the above equation with (2.15), we thus have

$$\int_{\mathbb{Z}_p} P_n(x) d\mu_1(x) = a_0 - \frac{1}{2}a_1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} a_{2j} B_{2j}.$$

Similarly, substituting

$$f(x, t; \lambda) = \lambda^x e^{tx}$$

into (2.5), we have

$$(2.17) \quad \int_{\mathbb{Z}_p} \lambda^x e^{t(x+y)} d\mu_1(x) = \frac{\log \lambda + t}{\lambda e^t - 1} e^{ty},$$

where $\lambda \in \mathbb{Z}_p$ (cf. [72]; see also [41], [124], [112]). Combining (2.17) with (1.25), we have

$$\int_{\mathbb{Z}_p} \lambda^x (x+y)^n d\mu_1(x) = \mathfrak{B}_n(y; \lambda)$$

According to [102], [131], [54] and [74], for each integer $N \geq 0$; C_{p^N} denotes the multiplicative group of the primitive p^N th roots of unity in $\mathbb{C}_p^* = \mathbb{C}_p \setminus \{0\}$.

Let

$$\mathbb{T}_p = \left\{ \xi \in \mathbb{C}_p : \xi^{p^N} = 1, \text{ for } N \geq 0 \right\} = \cup_{N \geq 0} C_{p^N}$$

In the sense of the p -adic Pontrjagin duality, the dual of \mathbb{Z}_p is $\mathbb{T}_p = C_{p^\infty}$, the direct limit of cyclic groups C_{p^N} of order p^N with $N \geq 0$, with the discrete topology. \mathbb{T}_p accept a natural \mathbb{Z}_p -module structure which can be written briefly as ξ^x for $\xi \in \mathbb{T}_p$ and $x \in \mathbb{Z}_p$. \mathbb{T}_p are embedded discretely in \mathbb{C}_p as the multiplicative p -torsion subgroup. If $\xi \in \mathbb{T}_p$, then $\vartheta_\xi : (\mathbb{Z}_p, +) \rightarrow (\mathbb{C}_p, \cdot)$ is the locally constant character, $x \rightarrow \xi^x$, which is a locally analytic character if $\xi \in \{\xi \in \mathbb{C}_p : \text{ord}_p(\xi - 1) > 0\}$. Consequently, it is well-known that ϑ_ξ has a continuation to a continuous group homomorphism from $(\mathbb{Z}_p, +)$ to (\mathbb{C}_p, \cdot) (cf. [102], [131], [54], [74], [103]; see also the references cited in each of these earlier works).

We assume that $\lambda \in \mathbb{T}_p$. Then we have

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = \mathcal{B}_n(\lambda) = \frac{nH_{n-1}(\lambda^{-1})}{\lambda - 1}$$

(cf. [72]).

The Volkenborn integral of some trigonometric functions are given as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} \cos(ax) d\mu_1(x) &= \frac{a \sin(a)}{2(1 - \cos(a))} \\ &= \frac{a}{2} \cot\left(\frac{a}{2}\right) \\ &= \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{a^{2n}}{(2n)!}, \end{aligned}$$

where $a \in \mathbb{E}$ with $a \neq 0$, $p \neq 2$ (cf. [101, p. 172], [60]);

$$\int_{\mathbb{Z}_p} \sin(ax) d\mu_1(x) = -\frac{a}{2},$$

where $a \in \mathbb{E}$ (cf. [101, p. 170], [60]); and also

$$\int_{\mathbb{Z}_p} \tan(ax) d\mu_1(x) = -\frac{a}{2}.$$

Note that

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_1(x) = \frac{1}{2} \int_{\mathbb{Z}_p} e^{ax} d\mu_1(x) - \frac{1}{2} \int_{\mathbb{Z}_p} e^{-ax} d\mu_1(x).$$

Combining the above equation with (2.14), we have

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_1(x) = \frac{1}{2} \frac{a}{e^a - 1} + \frac{1}{2} \frac{a}{e^{-a} - 1} = -\frac{a}{2}.$$

2.2. p -adic integral over subsets:

Let V be a compact open subset of \mathbb{Z}_p . Let $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$. Then we have

$$\int_V f(x) d\mu_1(x) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x),$$

where

$$g(x) = \begin{cases} f(x) & \text{if } x \in V \\ 0 & \text{if } x \in \mathbb{Z}_p \setminus V \end{cases}$$

(cf. [101, p. 174]).

2.3. p -adic integral over $j + p^n \mathbb{Z}_p$:

Let $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$. Then we have

$$(2.18) \quad \int_{j+p^n \mathbb{Z}_p} f(x) d\mu_1(x) = \int_{p^n \mathbb{Z}_p} f(j+x) d\mu_1(x) = \frac{1}{p^n} \int_{\mathbb{Z}_p} f(j+p^n x) d\mu_1(x)$$

(cf. [101, p. 175]). Substituting $f(x) = x^m$ with $m \in \mathbb{N}$ into (2.18), we have

$$\int_{j+p^n \mathbb{Z}_p} x^m d\mu_1(x) = p^{n(m-1)} B_m \left(\frac{j}{p^n} \right)$$

(cf. [101, p. 175]).

We now give some examples for the above formula:

Let

$$\mathbf{T}_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p$$

and $f : \mathbf{T}_p \rightarrow \mathbb{Q}_p$ and a C^1 -function and also $f(-x) = -f(x)$ with $x \in \mathbf{T}_p$. Thus we have

$$\int_{\mathbf{T}_p} f(x) d\mu_1(x) = 0.$$

Therefore

$$\int_{\mathbf{T}_p} \frac{1}{x} d\mu_1(x) = \int_{\mathbf{T}_p} \frac{1}{x^3} d\mu_1(x) = \int_{\mathbf{T}_p} \frac{1}{x^5} d\mu_1(x) = \cdots = \int_{\mathbf{T}_p} \frac{1}{x^{2n+1}} d\mu_1(x) = 0,$$

where $n \in \mathbb{N}$ (cf. [101, p. 175]) and

$$(2.19) \quad \int_{\mathbf{T}_p} x^j (x^{p-1})^s d\mu_1(x) = (j + (p-1)s) \zeta_{p,j}(s),$$

where $\zeta_{p,j}(s)$ denotes the p -adic zeta function, $|s|_p < p^{\frac{p-2}{p-1}}$, $s \neq -\frac{j}{p-1}$ and $j \in \{0, 1, \dots, p-2\}$, $p \neq 2$ (cf. [101, p. 187], [124]).

Substituting $s = n$ ($n \in \mathbb{N}_0$) into (2.19), we have following values of the p -adic zeta function:

$$\int_{\mathbf{T}_p} (x^{p-1})^n d\mu_1(x) = \left(1 - p^{n(p-1)-1}\right) \frac{B_{n(p-1)}}{n(p-1)},$$

$$\int_{\mathbf{T}_p} x^j (x^{p-1})^n d\mu_1(x) = \left(1 - p^{j-1+n(p-1)}\right) \frac{B_{j+n(p-1)}}{j+n(p-1)}$$

whereas for $n \in \{2, 4, 6, 8, \dots\}$, $j = 0$ and $p = 2$; and consequently we also have

$$\int_{\mathbf{T}_p} x^n d\mu_1(x) = (1 - 2^{n-1}) \frac{B_n}{n}$$

(cf. [101, p. 187], [124]).

2.4. p -adic Integral of the Falling Factorial

Kim et al. [49] defined Witt-type identities for the Daehee numbers of the first kind by the following p -adic integral representation as follows:

$$(2.20) \quad D_n = \int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x),$$

or equivalently

$$(2.21) \quad \int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x) = \sum_{l=0}^n S_1(n, l) B_l,$$

where $n \in \mathbb{N}_0$ (cf. [49]).

Kim et al. [49] defined the Daehee numbers of the second kind as follows:

$$(2.22) \quad \widehat{D}_n = \int_{\mathbb{Z}_p} t^{(n)} d\mu_1(t).$$

Kim et al. [49] defined the Daehee polynomials of the first and second kind, respectively, as follows:

$$(2.23) \quad D_n(x) = \int_{\mathbb{Z}_p} (x+t)_{(n)} d\mu_1(t)$$

and

$$(2.24) \quad \widehat{D}_n(x) = \int_{\mathbb{Z}_p} (x+t)^{(n)} d\mu_1(t),$$

where $n \in \mathbb{N}_0$.

Combining the following relation

$$x_{(n)} = n! \binom{x}{n},$$

and (2.12) with (2.20), we also have

$$(2.25) \quad \int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x) = \frac{(-1)^n n!}{n+1},$$

where $n \in \mathbb{N}_0$ (cf. [49]).

In [49], Kim et al. gave the following formula:

$$(2.26) \quad D_n = \frac{(-1)^n n!}{n+1}.$$

By using (2.12), we have

$$(2.27) \quad \begin{aligned} \int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_1(x) &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{x}{m} d\mu_1(x) \\ &= \sum_{m=1}^n (-1)^m \binom{n-1}{m-1} \frac{1}{m+1} \\ &= \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} \frac{1}{m+1} \end{aligned}$$

(cf. [49], [52], [107]). By using (2.27), we obtain

$$(2.28) \quad \int_{\mathbb{Z}_p} (x+n-1)_{(n)} d\mu_1(x) = n! \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} \frac{1}{m+1}.$$

3. Integral Formulas for the Volkenborn Integral

In [113], we gave the following interesting and new integral formulas for the Volkenborn integral including the falling factorial and the rising factorial with their identities and relations, the combinatorial sums, the special numbers such as the Bernoulli numbers of the first kind, the Stirling numbers and the Lah numbers.

Theorem 3.1 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(3.1) \quad \int_{\mathbb{Z}_p} xx_{(n)} d\mu_1(x) = (-1)^{n+1} \frac{n!}{n^2 + 3n + 2}$$

$$(3.2) \quad = \sum_{k=1}^n S_1(n, k-1) B_k + B_{n+1}.$$

Theorem 3.2 (cf. [113]). *Let $n \in \mathbb{N}$. Then we have*

$$(3.3) \quad \int_{\mathbb{Z}_p} xx^{(n)} d\mu_1(x) = \sum_{k=1}^n (-1)^{k+1} \binom{n-1}{k-1} \frac{n!}{k^2 + 3k + 2}$$

$$(3.4) \quad = \sum_{k=1}^n C(n, k) B_{k+1}.$$

Theorem 3.3 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \frac{x^{(n+1)}}{x} d\mu_1(x) = \sum_{k=0}^n (-1)^n n_{(n-k)} \frac{k!}{k+1}.$$

Corollary 3.4 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} (x+1)_{(n+1)} d\mu_1(x) = \frac{(-1)^n}{n+2} n!.$$

By applying the Volkenborn integral to (1.10), we get the following formula:

Corollary 3.5 (cf. [113]).

$$(3.5) \quad \int_{\mathbb{Z}_p} \binom{x+m}{n} d\mu_1(x) = \sum_{k=0}^n (-1)^k \binom{m}{n-k} \frac{1}{k+1}.$$

In [113], we also gave the following Volkenborn integral formulas:

By applying the Volkenborn integral with respect to x and y to (1.9), we have

$$(3.6) \quad \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} d\mu_1(x) d\mu_1(y) \\ = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_1(x) d\mu_1(y).$$

By combining the following identity with the above equation:

$$(3.7) \quad \binom{x+y}{n} = \frac{1}{n!} (x+y)_{(n)} = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x_{(k)} y_{(n-k)},$$

and using (2.20) and (2.25), we also get the following theorem:

Theorem 3.6 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(3.8) \quad \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_1(x) d\mu_1(y) = \sum_{k=0}^n (-1)^k \frac{1}{(k+1)(n-k+1)} \\ (3.9) \quad = \frac{1}{n!} \sum_{k=0}^n \sum_{j=0}^k \binom{k}{j} S_1(n, k) B_j B_{k-j}.$$

Lemma 3.7 (cf. [113]). *Let $n \in \mathbb{N}$. Then we have*

$$(3.10) \quad \int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = \frac{(-1)^{n+1}}{n^2+n}.$$

Theorem 3.8 (cf. [113]). *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} (x+1)_{(n)} d\mu_1(x) = (-1)^{n+1} \frac{n!}{n^2+n}.$$

Corollary 3.9 (cf. [113]). *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} \Delta x_{(n)} d\mu_1(x) = (-1)^{n+1} (n-1)!.$$

Theorem 3.10 (cf. [113]). *Let $n \in \mathbb{N}$. Then we have*

$$(3.11) \quad \int_{\mathbb{Z}_p} (-x)_{(n)} d\mu_1(x) = \sum_{k=1}^n (-1)^k \frac{k! L(n, k)}{k+1}$$

$$(3.12) \quad = \sum_{k=1}^n (-1)^{k+n} \binom{n-1}{k-1} \frac{n!}{k+1}.$$

Corollary 3.11 (cf. [113]). Let $n \in \mathbb{N}_0$. Then we have

$$(3.13) \quad \int_{\mathbb{Z}_p} \binom{x+1}{n+1} d\mu_1(x) = \frac{(-1)^n}{n^2 + 3n + 2}.$$

Theorem 3.12 (cf. [113]). Let $k \in \mathbb{N}_0$. Then we have

$$(3.14) \quad \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_1(x) d\mu_1(y) = \sum_{l,m=1}^k D_l D_m C_{l,m}^{(k)}$$

$$(3.15) \quad = \sum_{l,m=1}^k (-1)^{l+m} \frac{l!m!}{(l+1)(m+1)} C_{l,m}^{(k)}$$

$$(3.16) \quad = \sum_{m=0}^k S_1(k, m) (B_m)^2.$$

Theorem 3.13 (cf. [113]). Let $n \in \mathbb{N}$. Then we have

$$\int_{\mathbb{Z}_p} x \binom{x-2}{n-1} d\mu_1(x) = (-1)^n \sum_{k=1}^n \frac{k}{k+1}.$$

Theorem 3.14 (cf. [113]). Let $n \in \mathbb{N}_0$. Then we have

$$\int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_1(x) = (-1)^n H_n,$$

where H_n denotes the harmonic numbers given by

$$(3.17) \quad H_n = \sum_{k=0}^n \frac{1}{k+1}.$$

Theorem 3.15 (cf. [113]). Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have

$$\int_{\mathbb{Z}_p} \binom{mx}{n} d\mu_1(x) = \sum_{k=0}^n \frac{(-1)^k}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{mk-mj}{n}.$$

Theorem 3.16 (cf. [113]). Let $n, r \in \mathbb{N}_0$. Then we have

$$(3.18) \quad \int_{\mathbb{Z}_p} \binom{x}{n}^r d\mu_1(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r.$$

Remark 3.17. Substituting $r = 1$ into (3.18), since $\binom{k-j}{n} = 0$ if $k-j < n$, we arrive at the equation (2.12).

Theorem 3.18 (cf. [113]). Let $n \in \mathbb{N}$ with $n > 2$. Then we have

$$(3.19) \quad \int_{\mathbb{Z}_p} \left\{ x \binom{x-2}{n-1} + x(x-1) \binom{n-3}{n-2} \right\} d\mu_1(x) = (-1)^n \sum_{k=0}^n \frac{k^2}{k+1}.$$

Theorem 3.19 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(3.20) \quad \int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_1(x) = \sum_{k=0}^n \frac{(-1)^k}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j+n}{n}$$

$$(3.21) \quad = \sum_{k=0}^n B_k \sum_{j=0}^n \binom{n}{j} \frac{S_1(j, k)}{j!}.$$

Theorem 3.20 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+n+\frac{1}{2}}{n} d\mu_1(x) = \binom{2n}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{2^{2k-2n} (2n+1)}{(k+1)(2k+1) \binom{2k}{k}}.$$

Theorem 3.21 (cf. [113]). *Let $m, n \in \mathbb{N}_0$. Then we have*

$$(3.22) \quad \int_{\mathbb{Z}_p} x^m x_{(n)} d\mu_1(x) = \sum_{k=0}^n S_1(n, k) B_{k+m}.$$

Theorem 3.22 (cf. [113]). *Let $m, n \in \mathbb{N}_0$. Then we have*

$$(3.23) \quad \int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_1(x) = \sum_{j=0}^n \sum_{l=0}^m S_1(n, k) S_1(m, l) B_{j+l}$$

$$(3.24) \quad = \sum_{k=0}^m (-1)^{m+n-k} \binom{m}{k} \binom{n}{k} \frac{k!(m+n-k)!}{m+n-k+1}$$

$$(3.25) \quad = \sum_{k=0}^m \binom{m}{k} \binom{n}{k} k! \sum_{l=0}^{m+n-k} S_1(m+n-k, l) B_l$$

Let $z \in \mathbb{C}_p$, we have

$$(3.26) \quad D_m(z : q) = \int_{\mathbb{Z}_p} [x]^m d\mu_z(x)$$

(cf. [58]). If we take $z = q$ in (3.26), then we see that $D_m(q : q) = \beta_m(q)$, Carlitz's q -Bernoulli numbers. In the case when $z = u$ in (3.26), q -Daehee numbers and q -Daehee polynomials are defined, respectively, as follows:

$$(3.27) \quad D_m(u : q) = \int_{\mathbb{Z}_p} [x]^m d\mu_u(x) = H_m(u^{-1} : q)$$

and

$$D_m(z, x : q) = \int_{\mathbb{Z}_p} [x+t]^m d\mu_z(t),$$

where $m \in \mathbb{N}_0$ and $z \in \mathbb{C}_p$ (cf. [58], [124, Eq. (1.10)]).

Theorem 3.23. (cf. [121, Theorem 1]) Assume that a, b are integers with $(a, b) = (p, b) = 1$. Let

$$S_q(a, b : n; q^l) = \sum_{M=1}^{k-1} \frac{[M]}{[b]} \int_{\mathbb{Z}_p} q^{-lx} \left[x + \left\{ \frac{aM}{b} \right\} : q^l \right]^n d\mu_{q^l}(x).$$

Observe that

$$D_m(u : q) = H_m(u^{-1} : q),$$

(cf. [58, Eq. (5)]).

Corollary 3.24. (cf. [121, Corollary 1]) If $z = \lambda$, then we have

$$(3.28) \quad B_q(d, c : 0, \lambda : m) = \frac{m}{[c^m]} \sum_{\lambda} \frac{1}{[\lambda - 1][\lambda^{-d} - 1]} \int_{\mathbb{Z}_p} [t]^m d\mu_{\lambda}(t).$$

Remark 3.25. If $q \rightarrow 1$, then (3.28) is reduced to (3.29). That is,

$$\lim_{q \rightarrow 1} B_q(d, c : 0, \lambda : m) = S(d, c : m)$$

where

$$(3.29) \quad S(d, c : m) = \frac{m}{c^m} \sum_{\lambda} \frac{H_{m-1}(\lambda^{-1})}{(\lambda - 1)(\lambda^{-d} - 1)},$$

where λ runs through the c th roots of unity distinct from 1 and $H_m(\lambda)$ is the Frobenius-Euler numbers (cf. [121, Remark 1]).

Now, it is time to raise the following question:

Is it possible to give any reciprocity law for the q -Dedekind type sum $B_q(d, c : 0, \lambda : m)$. That is, how can we calculate the following relation:

$$B_q(d, c : 0, \lambda : m) + B_q(c, d : 0, \lambda : m) = ?$$

3.1. Some Properties of the Fermionic p -adic Integral

Here, we give some well-known properties of the fermionic p -adic integral.

Let $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$. Kim [60] gave the following integral equation for the fermionic p -adic integral on \mathbb{Z}_p :

$$(3.30) \quad \int_{\mathbb{Z}_p} E^n[f(x)] d\mu_{-1}(x) + (-1)^{n+1} \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) \\ = 2 \sum_{j=0}^{n-1} (-1)^{n-1-j} f(j),$$

where $n \in \mathbb{N}$.

Substituting $n = 1$ into (3.30), we have very useful integral equation, which is used to construct generating functions associated with Euler-type numbers and polynomials, given as follows:

$$(3.31) \quad \int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2f(0)$$

(cf. [60]).

By using (2.6) and (3.31), the well-known Witt's type formulas for the Euler numbers and polynomials of the first kind are given as follows, respectively:

$$(3.32) \quad E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x)$$

and

$$(3.33) \quad E_n(z) = \int_{\mathbb{Z}_p} (z+x)^n d\mu_{-1}(x),$$

where $n \in \mathbb{N}_0$ (cf. [39], [59]; see also the references cited in each of these earlier works).

Theorem 3.26. *Let $n \in \mathbb{N}_0$. Then we have*

$$(3.34) \quad \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}(x) = (-1)^n 2^{-n}.$$

Theorem 3.26 was proved by Kim et al. [52, Theorem 2.3].

Substituting $x_{(n)} = n! \binom{x}{n}$ into (3.34), we have the following well-known identity:

$$(3.35) \quad \int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x) = (-1)^n 2^{-n} n!$$

where $n \in \mathbb{N}_0$ (cf. [52]).

Recently, by using the fermionic p -adic integral on \mathbb{Z}_p , Kim et al. [52] defined the Changhee numbers of the first and the second kind, respectively, as follows:

$$(3.36) \quad Ch_n = \int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x)$$

and

$$(3.37) \quad \widehat{Ch}_n = \int_{\mathbb{Z}_p} x^{(n)} d\mu_{-1}(x),$$

where $n \in \mathbb{N}_0$.

For $n \in \mathbb{N}_0$, Kim et al. [52] gave the following formula for the Changhee numbers of the first kind:

$$(3.38) \quad Ch_n = (-1)^n 2^{-n} n!.$$

Kim et al. [52] also defined the Changhee polynomials of the first and the second, respectively, as follows:

$$(3.39) \quad Ch_n(x) = \int_{\mathbb{Z}_p} (x+t)_{(n)} d\mu_{-1}(t)$$

and

$$(3.40) \quad \widehat{Ch}_n(x) = \int_{\mathbb{Z}_p} (x+t)^{(n)} d\mu_{-1}(t).$$

Therefore, by using Theorem 3.26, we have

$$(3.41) \quad \begin{aligned} \int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_{-1}(x) &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{x}{m} d\mu_{-1}(x) \\ &= \sum_{m=1}^n (-1)^m \binom{n-1}{m-1} 2^{-m} \\ &= \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} 2^{-m} \end{aligned}$$

(cf. [52], [107], [49]). By using (3.41), we have

$$(3.42) \quad \int_{\mathbb{Z}_p} (x+n-1)_{(n)} d\mu_{-1}(x) = n! \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} 2^{-m}.$$

By using (3.30), Kim [61] modified (2.6). He gave the following integral equation:

$$(3.43) \quad q^d \int_{\mathbb{Z}_p} E^d f(x) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2] \sum_{j=0}^{d-1} (-1)^j q^j f(j),$$

where d is an positive odd integer.

Some examples for the fermionic p -adic integral are given as follows:

The Volkenborn integral of some trigonometric functions are given as follows:

$$\int_{\mathbb{Z}_p} \cos(ax) d\mu_{-1}(x) = 1,$$

where $a \in \mathbb{E}$ with $a \neq 0$, $p \neq 2$ (cf. [60]);

$$\int_{\mathbb{Z}_p} \sin(a(x+1)) d\mu_{-1}(x) = - \int_{\mathbb{Z}_p} \sin(ax) d\mu_{-1}(x),$$

where $a \in \mathbb{E}$ (cf. [60]); and also

$$\int_{\mathbb{Z}_p} \sin(ax) d\mu_{-1}(x) = - \frac{\sin(a)}{\cos(a) + 1}.$$

Note that

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_{-1}(x) = \frac{1}{2} \int_{\mathbb{Z}_p} e^{ax} d\mu_{-1}(x) - \frac{1}{2} \int_{\mathbb{Z}_p} e^{-ax} d\mu_{-1}(x).$$

Combining the above equation with (3.31), we have

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_{-1}(x) = \frac{1}{e^a + 1} + \frac{1}{e^{-a} + 1} = 1.$$

Let

$$P_n(x) = \sum_{j=0}^n a_j x^j$$

be a polynomial of degree n ($n \in \mathbb{N}_0$). Substituting $P_n(x)$ into (2.6), we have

$$\int_{\mathbb{Z}_p} P_n(x) d\mu_{-1}(x) = \sum_{j=0}^n a_j \int_{\mathbb{Z}_p} x^j d\mu_{-1}(x).$$

Since $E_{2n} = 0$ for $n \in \mathbb{N}$, by combining the above equation with (3.32), we thus have

$$\int_{\mathbb{Z}_p} P_n(x) d\mu_{-1}(x) = 1 + \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} a_{2j+1} E_{2j+1}.$$

By using (2.3), we have

$$(3.44) \quad \int_{\mathbb{X}} d\mu_{\mathcal{E},k,\lambda}(x + fp^N \mathbb{Z}_p) = \mathcal{E}_k(\lambda),$$

where $\lambda \in \mathbb{Z}_p$ (cf. [88], [87]).

Substituting

$$g(x, t; \lambda) = \lambda^x e^{tx}$$

into (3.31), we have the following well-known formula:

$$\int_{\mathbb{Z}_p} g(x, t; \lambda) d\mu_{-1}(x) = \frac{2}{\lambda e^t + 1}.$$

Combining the above equation with (1.26), we have

$$(3.45) \quad \int_{\mathbb{Z}_p} \lambda^x x^n d\mu_{-1}(x) = \mathcal{E}_n(\lambda).$$

By assuming that χ is the primitive Dirichlet's character with odd conductor f , Rim and Kim [95] gave the following formula:

$$\int_{\mathbb{X}} \chi(x) d\mu_{E,k}(x + fp^N \mathbb{Z}_p) = E_{k,\chi}$$

where

$$\frac{2}{e^{ft} + 1} \sum_{j=0}^{f-1} (-1)^j \chi(j) e^{tj} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}.$$

Combining (3.44) with (3.45), we have the following well-known relation:

$$(3.46) \quad d\mu_{\mathcal{E},k,\lambda}(x + fp^N \mathbb{Z}_p) = \lambda^x x^n d\mu_{-1}(x + p^N \mathbb{Z}_p).$$

Setting $\lambda = 1$ in (3.46), we have

$$d\mu_{E,k}(x + fp^N \mathbb{Z}_p) = x^n d\mu_{-1}(x + p^N \mathbb{Z}_p)$$

or equivalently

$$d\mu_{E,k}(x) = x^n d\mu_{-1}(x)$$

(cf. [95]).

Therefore, combining (1.28) with (3.46), we have

$$d\mu_{\mathcal{E},k,\lambda}(x + fp^N \mathbb{Z}_p) = -\frac{2}{k+1} d\mu_{\mathcal{B},k,-\lambda}(x + fp^N \mathbb{Z}_p).$$

4. Integral Formulas for the Fermionic p -adic Integral

In [113], we gave the following interesting and new integral formulas for the fermionic p -adic integral including the falling factorial and the rising factorial with their identities and relations, the combinatorial sums, the special numbers such as the Euler numbers, the Stirling numbers and the Lah numbers.

Theorem 4.1 (cf. [113]). *Let $n \in \mathbb{N}$. Then we have*

$$(4.1) \quad \int_{\mathbb{Z}_p} x x_{(n)} d\mu_{-1}(x) = (-1)^n \frac{(n-1)}{2^{n+1}} n!.$$

Theorem 4.2 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(4.2) \quad \int_{\mathbb{Z}_p} x x^{(n)} d\mu_{-1}(x) = \sum_{k=1}^n (-1)^k \binom{n-1}{k-1} \frac{(k-1)}{2^{k+1}} n!.$$

Theorem 4.3 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(4.3) \quad \int_{\mathbb{Z}_p} (x+1)_{(n)} d\mu_{-1}(x) = (-1)^{n+1} \frac{1}{2^n} n!.$$

Theorem 4.4 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \frac{x^{(n+1)}}{x} d\mu_{-1}(x) = \sum_{k=0}^n (-1)^n n_{(n-k)} \frac{k!}{2^k}.$$

Theorem 4.5 (cf. [113]). *Let $k \in \mathbb{N}_0$. Then we have*

$$(4.4) \quad \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_{-1}(x) d\mu_{-1}(y) = \sum_{l,m=1}^k (-1)^{l+m} 2^{-m-l} l! m! C_{l,m}^{(k)}$$

$$(4.5) \quad = \sum_{m=0}^k S_1(k, m) (E_m)^2.$$

Theorem 4.6 (cf. [113]). *Let $n \in \mathbb{N}$ with $n > 2$. Then we have*

$$\int_{\mathbb{Z}_p} \left\{ x \binom{x-2}{n-1} + x(x-1) \binom{n-3}{n-2} \right\} d\mu_{-1}(x) = (-1)^n \sum_{k=0}^n \frac{k^2}{2^k}.$$

Theorem 4.7 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(4.6) \quad \int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_{-1}(x) = \sum_{k=0}^n \frac{(-1)^k}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j+n}{n}$$

$$(4.7) \quad = \sum_{k=0}^n E_k \sum_{j=0}^n \binom{n}{j} \frac{S_1(j, k)}{j!}.$$

Theorem 4.8 (cf. [113]). *Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{mx}{n} d\mu_{-1}(x) = \sum_{k=0}^n \frac{(-1)^k}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{mk-mj}{n}.$$

Theorem 4.9 (cf. [113]). *Let $n, r \in \mathbb{N}_0$. Then we have*

$$(4.8) \quad \int_{\mathbb{Z}_p} \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r.$$

Theorem 4.10 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_{-1}(x) = \sum_{k=1}^n 2^{-k}.$$

Theorem 4.11 (cf. [113]). *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+n+\frac{1}{2}}{n} d\mu_{-1}(x) = (2n+1) \binom{2n}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{2^{k-2n}}{(2k+1) \binom{2k}{k}}.$$

Theorem 4.12 (cf. [66]). *Let $n \in \mathbb{N}_0$. Then we have*

$$(4.9) \quad \int_{\mathbb{Z}_p} (1-x)^n d\mu_{-1}(x) = 2 + \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x).$$

Theorem 4.13 (cf. [66]). Let $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$. If $k = 0$, we have

$$(4.10) \quad \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = 2 + E_n$$

and if $k > 0$, we have

$$(4.11) \quad \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} E_{n-j}.$$

In [69], Kim et al. gave the following formula:

$$(4.12) \quad \int_{\mathbb{Z}_p} (-y^w (e^t - 1)^w)^x d\mu_{-1}(x) = \frac{2}{1 - y^w (e^t - 1)^w},$$

where $w \in \mathbb{N}$. By using the above formula, they defined so-called w -torsion Fubini polynomials. If $w = y = 1$, right-hand side of the equation (4.12) reduces to generating function for the Fubini numbers (cf. [46]).

5. New Integral Formulas Involving Volkenborn Integral

In this section, we give some new integral formulas for the Volkenborn integral. These new formulas are related to some special functions, special numbers and polynomials such as rising factorial and the falling factorial, the Bernoulli numbers and polynomials of the first kind, the Euler numbers and polynomials of the first kind, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

Theorem 5.1. Let $m, n \in \mathbb{N}_0$. Then we have

$$(5.1) \quad \int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_1(x) = (-1)^{m+n} \frac{(m+n)!}{m+n+1}.$$

Proof. By applying the Volkenborn integral to following well-known identity:

$$(5.2) \quad x_{(m+n)} = x_{(m)} (x - m)_{(n)},$$

we get

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_1(x) = \int_{\mathbb{Z}_p} x_{(m+n)} d\mu_1(x).$$

Combining the above equation with (2.25), we get the desired result. \square

Theorem 5.2. Let $n \in \mathbb{N}_0$. Then we have

$$(5.3) \quad \int_{\mathbb{Z}_p} x^{[n]} d\mu_1(x) = \sum_{k=0}^n t(n, k) B_k.$$

Proof. By applying the Volkenborn integral to the equation (1.56), we get

$$\int_{\mathbb{Z}_p} x^{[n]} d\mu_1(x) = \sum_{k=0}^n t(n, k) \int_{\mathbb{Z}_p} x^k d\mu_1(x).$$

Combining the above equation with (2.15), we arrive at the desired result. \square

Theorem 5.3. *Let $n \in \mathbb{N}$ with $n \geq 2$. Then we have*

$$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_1(x) = \sum_{k=0}^n t(n, k) B_k + \left(\frac{n-2}{2}\right)^2 \sum_{k=0}^{n-2} t(n-2, k) B_k.$$

Proof. By applying the Volkenborn integral to the following well-known equation

$$(5.4) \quad x^{[n]} = \left(x^2 - \left(\frac{n-2}{2}\right)^2\right) x^{[n-2]}$$

(cf. [24, p. 11]), we get

$$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_1(x) = \int_{\mathbb{Z}_p} x^{[n]} d\mu_1(x) + \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{Z}_p} x^{[n-2]} d\mu_1(x).$$

Combining the above equation with (5.3), we arrive at the desired result. \square

Theorem 5.4. *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_1(x) = \sum_{k=0}^{2n} t(2n, k) B_{2k}.$$

Proof. By applying the Volkenborn integral to the following well-known equation

$$(5.5) \quad x^{[2n]} = x^2 \prod_{k=1}^{n-1} (x^2 - k^2),$$

which is an even function (cf. [24, Eq. (2.1)]), we get

$$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_1(x) = \int_{\mathbb{Z}_p} x^{[2n]} d\mu_1(x).$$

Combining right-hand side of the above equation with (5.3), we arrive at the desired result. \square

Theorem 5.5. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x \prod_{k=1}^n \left(x^2 - \frac{(2k-1)^2}{4}\right) d\mu_1(x) = -\frac{1}{2} \frac{d}{dx} \left\{ x^{[2n+1]} \right\} \Big|_{x=0}$$

Proof. By applying the Volkenborn integral to the following well-known equation

$$(5.6) \quad x^{[2n+1]} = x \prod_{k=1}^n \left(x^2 - \frac{(2k-1)^2}{4} \right),$$

which is an odd function (cf. [24, Eq. (2.2)]), we get

$$(5.7) \quad \int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_1(x) = \int_{\mathbb{Z}_p} x^{[2n+1]} d\mu_1(x).$$

Since the function $x^{[2n+1]}$ is an odd function, combining right-hand side of the equation (5.7) with (2.13), we arrive at the desired result. \square

Remark 5.6. By combining (5.1) with (2.20) and (2.21), we get the following identities:

$$\int_{\mathbb{Z}_p} x_{(m)} (x-m)_{(n)} d\mu_1(x) = D_{m+n}$$

and

$$\int_{\mathbb{Z}_p} x_{(m)} (x-m)_{(n)} d\mu_1(x) = \sum_{k=0}^{n+m} S_1(m+n, k) B_k.$$

Theorem 5.7. *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} x \binom{x-2}{n-1} d\mu_1(x) = (-1)^{-n} \sum_{k=1}^n \frac{k}{k+1}.$$

Proof. By applying the Volkenborn integral to (1.14), and using (2.12), we get the desired result. \square

Theorem 5.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} \int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_1(x) &= \sum_{k=1}^n \frac{1}{k+1} \\ &= H_n - H_0. \end{aligned}$$

Proof. By applying the Volkenborn integral to (1.15), and using (2.12) and (3.17), we get the desired result. \square

Theorem 5.9. *Let $n, r \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_1(x) = \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \sum_{l=0}^k \frac{S_1(k, l) B_{v+l}}{k!}.$$

Proof. By applying the Volkenborn integral to (1.17), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_1(x) &= \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \int_{\mathbb{Z}_p} x^v \binom{x}{k} d\mu_1(x) \\ &= \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \int_{\mathbb{Z}_p} \frac{x^v}{k!} x^{(k)} d\mu_1(x). \end{aligned}$$

Combining the above equation with (1.34), we obtain

$$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_1(x) = \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \sum_{l=0}^k \frac{S_1(k, l)}{k!} \int_{\mathbb{Z}_p} x^{v+l} d\mu_1(x).$$

Combining the above equation with (2.15), we arrive at the desired result. \square

Theorem 5.10. *Let $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$. Then we have*

$$\sum_{k=0}^n (-1)^{k-n} \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) = \sum_{j=0}^n \binom{n}{j} (-2)^{n-j} B_{n-j}.$$

Proof. By applying the Volkenborn integral to following well-known identity:

$$(5.8) \quad \sum_{k=0}^n (-1)^k B_k^n(x) = (1-2x)^n$$

(cf. [108, Theorem 3.4]), we get

$$\begin{aligned} (5.9) \quad \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) &= \int_{\mathbb{Z}_p} (1-2x)^n d\mu_1(x) \\ &= \sum_{j=0}^n \binom{n}{j} (-2)^{n-j} \int_{\mathbb{Z}_p} x^{n-j} d\mu_1(x). \end{aligned}$$

By combining (5.9) with (2.15), we get the desired result. \square

Combining (5.9) with (1.48), we obtain

$$\begin{aligned} (5.10) \quad \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) &= \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-2)^{n-j} S_2(n-j, m) \int_{\mathbb{Z}_p} x_{(m)} d\mu_1(x). \end{aligned}$$

Combining (5.10) with (2.25), we arrive at the following theorem:

Theorem 5.11. *Let $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$. Then we have*

$$(5.11) \quad \begin{aligned} & \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) \\ &= \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-1)^{n+m-j} 2^{n-j} S_2(n-j, m) \frac{m!}{m+1}. \end{aligned}$$

Combining (5.10) with (2.20), we arrive at the following result:

Corollary 5.12. *Let $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$. Then we have*

$$(5.12) \quad \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-2)^{n-j} S_2(n-j, m) D_m.$$

By applying the Volkenborn integral to (1.64) and (1.66), using (2.12), (2.20) (2.25) and (2.21), we arrive at the following results:

Theorem 5.13. *Let $n \in \mathbb{N}_0$. Then we have*

$$(5.13) \quad \int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_1(x) = \sum_{v=0}^n \binom{n}{v} s_v(\lambda, \mu) D_{n-v}.$$

Theorem 5.14.

$$(5.14) \quad \int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_1(x) = \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} \frac{s_v(\lambda, \mu)(n-v+1)!}{n-v}.$$

Theorem 5.15.

$$(5.15) \quad \int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_1(x) = \sum_{v=0}^n \binom{n}{v} s_v(\lambda, \mu) \sum_{l=0}^{n-v} S_1(n-v, l) B_l.$$

Combining (5.15) with (2.25), we also have the following theorem:

Theorem 5.16.

$$(5.16) \quad \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_1(x) = (-1)^n \frac{n!}{n+1}.$$

Theorem 5.17. *Let $H_k \in \mathbb{H}$, the set of harmonic numbers. Let $1 \leq n \leq k$. Then we have*

$$\int_{\mathbb{Z}_p} \prod_{j=1}^k (1+jx) d\mu_1(x) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} B_n,$$

where $\binom{H_k}{n}_{\mathbb{H}}$ denotes the harmonic binomial coefficient.

Proof. In [23, Theorem 3.17], Brigham II defined the following identity:

$$(5.17) \quad \prod_{j=1}^k (1 + jx) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} x^n,$$

where $\binom{H_k}{n}_{\mathbb{H}}$ denotes the harmonic binomial coefficient, which given in [23, Lemma 3.2] as follows:

$$\binom{H_k}{n}_{\mathbb{H}} = \frac{1}{k!} \left[\begin{matrix} k+1 \\ n+1 \end{matrix} \right]$$

in which $\left[\begin{matrix} k \\ n \end{matrix} \right]$ denotes the unsigned Stirling numbers of the first kind. By applying the Volkenborn integral to (5.17), we get

$$\int_{\mathbb{Z}_p} \prod_{j=1}^k (1 + jx) d\mu_1(x) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} \int_{\mathbb{Z}_p} x^n d\mu_1(x).$$

Combining the above equation with (2.15), we arrive at the desired result. \square

6. New Integral Formulas Involving Fermionic p -adic Integral

In this section, we give some new integral formulas for the fermionic p -adic integral. These new formulas are related to some special functions, special numbers and polynomials such as rising factorial and the falling factorial, the Bernoulli numbers and polynomials of the first kind, the Euler numbers and polynomials of the first kind, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Dae-hee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

By applying the fermionic p -adic integral to (1.34), and using (3.32), we get the following theorem:

Theorem 6.1. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$(6.1) \quad \int_{\mathbb{Z}_p} x^m x_{(n)} d\mu_{-1}(x) = \sum_{k=0}^n S_1(n, k) E_{k+m}.$$

Theorem 6.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$(6.2) \quad \int_{\mathbb{Z}_p} x^{[n]} d\mu_{-1}(x) = \sum_{k=0}^n t(n, k) E_k.$$

Proof. By applying the fermionic p -adic integral to the equation (1.56), we get

$$\int_{\mathbb{Z}_p} x^{[n]} d\mu_{-1}(x) = \sum_{k=0}^n t(n, k) \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x).$$

Combining the above equation with (3.32), we arrive at the desired result. \square

Theorem 6.3. *Let $n \in \mathbb{N}$ with $n \geq 2$. Then we have*

$$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_{-1}(x) = \sum_{k=0}^n t(n, k) E_k + \left(\frac{n-2}{2}\right)^2 \sum_{k=0}^{n-2} t(n-2, k) E_k.$$

Proof. By applying the fermionic p -adic integral to the equation (5.4), we get

$$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x^{[n]} d\mu_{-1}(x) + \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{Z}_p} x^{[n-2]} d\mu_{-1}(x).$$

Combining the above equation with (6.2), we arrive at the desired result. \square

Corollary 6.4. *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_{-1}(x) = \sum_{k=0}^{2n} t(2n, k) E_{2k}.$$

Proof. By applying the Volkenborn integral to the equation (5.5), we get

$$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x^{[2n]} d\mu_{-1}(x).$$

Combining right-hand side of the above equation with (6.2), we arrive at the desired result. \square

Theorem 6.5. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$(6.3) \quad \int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_{-1}(x) = (-1)^{m+n} \frac{(m+n)!}{2^{m+n}}.$$

Proof. By applying the fermionic p -integral to (5.2), we obtain

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x_{(m+n)} d\mu_{-1}(x).$$

Combining the above equation with (3.35), we get the desired result. \square

Remark 6.6. Combining (6.3) with (3.36) and (1.62), we arrive at the following identities:

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_{-1}(x) = Ch_{m+n}$$

and

$$\int_{\mathbb{Z}_p} x_{(m)} (x-m)_{(n)} d\mu_{-1}(x) = \sum_{k=0}^{n+m} S_1(n+m, k) E_k.$$

By applying the fermionic p -adic integral to (1.8), and using (3.35), we get the following theorem:

Theorem 6.7. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$(6.4) \quad \int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_{-1}(x) = \sum_{k=0}^m (-1)^{m+n-k} \binom{m}{k} \binom{n}{k} \frac{k!(m+n-k)!}{2^{m+n-k}}.$$

Theorem 6.8. *Let $n, r \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \sum_{j=0}^k \sum_{l=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \frac{S_1(k, l) E_{v+l}}{k!}.$$

Proof. By applying the fermionic p -adic integral to (1.17), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_{-1}(x) &= \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \int_{\mathbb{Z}_p} x^v \binom{x}{k} d\mu_{-1}(x) \\ &= \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \int_{\mathbb{Z}_p} \frac{x^v}{k!} x_{(k)} d\mu_{-1}(x). \end{aligned}$$

Combining the above equation with (1.34), we obtain

$$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \sum_{l=0}^k \frac{S_1(k, l)}{k!} \int_{\mathbb{Z}_p} x^{v+l} d\mu_{-1}(x).$$

Combining the above equation with (3.32), we arrive at the desired result. \square

Theorem 6.9. *Let $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$. Then we have*

$$(6.5) \quad \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \sum_{j=0}^n \binom{n}{j} (-2)^{n-j} E_{n-j}.$$

Proof. By applying the fermionic p -integral to (5.8), we obtain

$$\begin{aligned} (6.6) \quad \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} (1-2x)^n d\mu_{-1}(x) \\ &= \sum_{j=0}^n \binom{n}{j} (-2)^{n-j} \int_{\mathbb{Z}_p} x^{n-j} d\mu_{-1}(x). \end{aligned}$$

Combining the above equation with (3.32), we arrive at the desired result. \square

Theorem 6.10. *Let $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$. Then we have*

$$\sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-1)^{m+n-j} 2^{n-j-m} S_2(n-j, m) m!.$$

Proof. Combining (6.6) with (1.48), we have

$$\begin{aligned} (6.7) \quad & \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) \\ &= \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-2)^{n-j} S_2(n-j, m) \int_{\mathbb{Z}_p} x_{(m)} d\mu_{-1}(x). \end{aligned}$$

Combining (6.7) with (3.35), we get the desired result. \square

Combining (6.7) with (3.36), we arrive at the following corollary:

Corollary 6.11. *Let $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$. Then we have*

$$(6.8) \quad \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-2)^{n-j} S_2(n-j, m) Ch_m.$$

By applying the fermionic p -integral to (1.64) and (1.66), using (3.35) and (3.36), we arrive at the following theorems, respectively:

Theorem 6.12. *Let $n \in \mathbb{N}_0$. Then we have*

$$(6.9) \quad \int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_{-1}(x) = \sum_{v=0}^n \binom{n}{v} s_v(\lambda, \mu) Ch_{n-v}.$$

Theorem 6.13. *Let $n \in \mathbb{N}_0$. Then we have*

$$(6.10) \quad \int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_{-1}(x) = \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} \frac{s_v(\lambda, \mu) (n-v)!}{2^{n-v}}.$$

Theorem 6.14. *Let $n \in \mathbb{N}_0$. Then we have*

$$(6.11) \quad \int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x) = \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x).$$

Theorem 6.15. *Let $n, \mu \in \mathbb{N}_0$. Then we have*

$$(6.12) \quad \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x) = (-1)^n \frac{n!}{2^n}.$$

By applying the fermionic p -integral to (1.67), we have

$$\sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x).$$

Combining the above equation with (3.35) and (3.36), we obtain the following results:

Theorem 6.16. *Let $n, v \in \mathbb{N}_0$. Then we have*

$$(6.13) \quad \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x) = Ch_n.$$

Theorem 6.17. *Let $n, v \in \mathbb{N}_0$. Then we have*

$$(6.14) \quad \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x) = (-1)^n \frac{n!}{2^n}.$$

Theorem 6.18. *Let $H_k \in \mathbb{H}$, the set of harmonic numbers. Let $1 \leq n \leq k$. Then we have*

$$\int_{\mathbb{Z}_p} \prod_{j=1}^k (1 + jx) d\mu_{-1}(x) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} E_n,$$

where $\binom{H_k}{n}_{\mathbb{H}}$ denotes the harmonic binomial coefficient.

Proof. By applying the fermionic p -adic integral to (5.17), we get

$$\int_{\mathbb{Z}_p} \prod_{j=1}^k (1 + jx) d\mu_{-1}(x) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x).$$

Combining the above equation with (3.32), we arrive at the desired result. \square

7. Identities and Relations

By using the results obtained in the previous sections, we give some new formulas and relations in this section. These formulas and relations are involving the Bernoulli numbers and polynomials of the first kind, the Euler numbers and polynomials of the first kind, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

Theorem 7.1. *Let $l \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have*

$$(7.1) \quad \sum_{j=0}^n \binom{n}{j} \frac{B_{j+l}}{j+l} = \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \left(\frac{B_{n+k}(1) - B_0}{n+k} \right).$$

Proof. By applying the Volkenborn integral to the following well-known combinatorial series identity:

$$(7.2) \quad \sum_{j=0}^n \binom{n}{j} \frac{x^{j+l}}{j+l} = \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \left(\frac{(1+x)^{n+k} - 1}{n+k} \right),$$

(cf. [17, Eq. (2.4)]), we obtain

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} \frac{1}{j+l} \int_{\mathbb{Z}_p} x^{j+l} d\mu_1(x) \\ &= \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \frac{1}{n+k} \int_{\mathbb{Z}_p} \left((1+x)^{n+k} - 1 \right) d\mu_1(x). \end{aligned}$$

Combining the above equation with (2.15) and (2.16), we arrive at the desired result. \square

For $l = 1$, (7.1) coincides with the following corollary:

Corollary 7.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} \frac{B_{j+1}}{j+1} = \frac{B_{n+1}(1) - B_0}{n+1}.$$

Theorem 7.3. *Let $l \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have*

$$(7.3) \quad \sum_{j=0}^n \binom{n}{j} \frac{E_{j+l}}{j+l} = \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \left(\frac{E_{n+k}(1) - E_0}{n+k} \right).$$

Proof. That is, by applying the fermionic p -adic integral (7.2), we obtain

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} \frac{1}{j+l} \int_{\mathbb{Z}_p} x^{j+l} d\mu_{-1}(x) \\ &= \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \frac{1}{n+k} \int_{\mathbb{Z}_p} \left((1+x)^{n+k} - 1 \right) d\mu_{-1}(x). \end{aligned}$$

Combining the above equation with (3.32) and (3.33), we arrive at the desired result. \square

For $l = 1$, (7.3) coincides with the following corollary:

Corollary 7.4. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} \frac{E_{j+1}}{j+1} = \frac{E_{n+1}(1) - E_0}{n+1}.$$

Remark 7.5. By using (7.2), Choi and Srivastava [17, Lemma 1] gave the following summation formulas involving harmonic numbers and combinatorial series identity:

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{j+l} = \frac{1}{n+1},$$

where $n \in \mathbb{N}_0$ and

$$\sum_{j=0}^n (-1)^{j+1} \binom{n}{j} \frac{H_j}{j+l} = \frac{H_n}{n+1},$$

where $n \in \mathbb{N}_0$.

Theorem 7.6. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$(7.4) \quad B_n = \sum_{k=0}^n \sum_{j=0}^k T(n, k) t(k, j) B_j.$$

Proof. By applying the Volkenborn integral to the equation (1.57), we get

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = \sum_{k=0}^n T(n, k) \int_{\mathbb{Z}_p} x^{[k]} d\mu_1(x).$$

Combining the above equation with (2.15) and (5.3), we arrive at the desired result. \square

Theorem 7.7. *Let $n \in \mathbb{N}_0$. Then we have*

$$(7.5) \quad E_n = \sum_{k=0}^n \sum_{j=0}^k T(n, k) t(k, j) E_j.$$

Proof. By applying the Volkenborn integral to the equation (1.57), we get

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \sum_{k=0}^n T(n, k) \int_{\mathbb{Z}_p} x^{[k]} d\mu_{-1}(x).$$

Combining the above equation with (3.32) and (6.2), we arrive at the desired result. \square

By using (6.5), we have

$$\int_{\mathbb{Z}_p} B_0^n(x) d\mu_{-1}(x) + \sum_{k=1}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \sum_{j=0}^n (-2)^{n-j} E_{n-j}.$$

Combining the above equation with (4.10) and (4.11), we arrive at the following theorem:

Theorem 7.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$E_n = \sum_{j=0}^n (-2)^{n-j} E_{n-j} - \sum_{k=1}^n (-1)^k \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} E_{n-j} - 2.$$

Combining (6.5) with (6.8), we get the following result:

Theorem 7.9. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} (-2)^{n-j} \left(E_{n-j} - \sum_{m=0}^{n-j} S_2(n-j, m) Ch_m \right) = 0.$$

Combining (5.11) with (5.12) and (2.21), we arrive at the following results:

Theorem 7.10. *Let $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} & \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-1)^{n+m-j} \frac{2^{n-j} S_2(n-j, m) m!}{m+1} \\ &= \sum_{j=0}^n \sum_{m=0}^{n-j} \sum_{l=0}^m \binom{n}{j} (-2)^{n-j} S_2(n-j, m) S_1(m, l) B_l. \end{aligned}$$

By applying the Volkenborn integral to (1.67), we have

$$\sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_1(x) = \int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x).$$

Combining the above equation with (2.12), (2.20), (2.25), (2.26) and (2.21), we arrive at the following theorems, respectively:

Theorem 7.11. *Let $n, v \in \mathbb{N}_0$. Then we have*

$$D_n = \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) \sum_{m=0}^{n-v} \binom{n-v}{m} s_m(\lambda, \mu) \sum_{l=0}^{n-v-m} S_1(n-v-m, l) B_l.$$

Theorem 7.12. *Let $n, v \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} & \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) \sum_{m=0}^{n-v} \binom{n-v}{m} s_m(\lambda, \mu) \sum_{l=0}^{n-v-m} S_1(n-v-m, l) B_l \\ &= (-1)^n \frac{n!}{n+1}. \end{aligned}$$

Theorem 7.13. *Let $n, v \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} & \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) \sum_{m=0}^{n-v} \binom{n-v}{m} s_m(\lambda, \mu) \sum_{l=0}^{n-v-m} S_1(n-v-m, l) B_l \\ &= \sum_{v=0}^n S_1(n, v) B_l. \end{aligned}$$

By combining (5.13), (5.14), (5.15) and (5.16) with (2.25), (2.26) and (2.21), we get the following results:

Theorem 7.14. *Let $n, \mu \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} \binom{n-v}{l} s_l(\lambda, \mu) D_{n-v-l} = (-1)^n \frac{n!}{n+1}.$$

Theorem 7.15. *Let $n, \mu \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} (-1)^{n-v-l} \binom{n-v}{l} \frac{s_l(\lambda, \mu)(n-v-l)!}{n-v-l+1} = (-1)^n \frac{n!}{n+1}.$$

Combining (6.9) and (6.13), we arrive at the following results:

Theorem 7.16. *Let $n, v \in \mathbb{N}_0$. Then we have*

$$(7.6) \quad Ch_n = \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) \sum_{m=0}^{n-v} \binom{n-v}{m} s_m(\lambda, \mu) Ch_{n-v-m}.$$

Combining (3.38) with (7.6), we arrive at the following corollary:

Corollary 7.17. *Let $n, v \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) \sum_{m=0}^{n-v} (-1)^{n-v-m} \binom{n-v}{m} \frac{s_m(\lambda, \mu)(n-v-m)!}{2^{n-v-m}} = (-1)^n \frac{n!}{2^n}.$$

After comparing and combining the equation (6.9) with the equation (6.12), and making the necessary algebraic operations, we obtain the following results, respectively:

Theorem 7.18. *Let $n, \mu \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} \binom{n-v}{l} s_l(\lambda, \mu) Ch_{n-v-l} = (-1)^n \frac{n!}{2^n}.$$

Theorem 7.19. *Let $n, \mu \in \mathbb{N}_0$. Then we have*

$$(7.7) \quad Ch_n = \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} (-1)^{n-v-l} \binom{n-v}{l} \frac{s_l(\lambda, \mu)(n-v-l)!}{2^{n-v-l}}.$$

Combining (3.38) with (7.7), we arrive at the following corollary:

Corollary 7.20. *Let $n, \mu \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} (-1)^{n-v-l} \binom{n-v}{l} \frac{s_l(\lambda, \mu)(n-v-l)!}{2^{n-v-l}} = (-1)^n \frac{n!}{2^n}.$$

Theorem 7.21. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) Ch_{n-j} = \sum_{j=0}^n (-1)^n j!(n-j)! \binom{n}{j} \frac{\lambda^{n+j}}{2^n (\lambda-1)^{j+1}}.$$

Proof. By applying the fermionic p -adic integral to (1.69) and (1.71), we have the following relations, respectively:

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_{-1}(x) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) \int_{\mathbb{Z}_p} x_{(n-j)} d\mu_{-1}(x)$$

and

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_{-1}(x) = 2 \sum_{j=0}^n (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j}}{(2\lambda - 2)^{j+1}} \int_{\mathbb{Z}_p} x_{(n-j)} d\mu_{-1}(x).$$

Combining the above equations with (3.36) and (3.35), we get:

$$(7.8) \quad \int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) Ch_{n-j}$$

and

$$(7.9) \quad \int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = \sum_{j=0}^n (-1)^n j!(n-j)! \binom{n}{j} \frac{\lambda^{n+j}}{2^n (\lambda - 1)^{j+1}}.$$

Combining (7.8) with (7.9), we arrive at the desired result. \square

Theorem 7.22. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) D_{n-j} = 2 \sum_{j=0}^n \sum_{l=0}^{n-j} (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j} S_1(n-j, l) B_l}{(2\lambda - 2)^{j+1}}.$$

Proof. By applying the Volkenborn integral to (1.69) and (1.71), we have the following relations, respectively:

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) \int_{\mathbb{Z}_p} x_{(n-j)} d\mu_1(x)$$

and

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = 2 \sum_{j=0}^n (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j}}{(2\lambda - 2)^{j+1}} \int_{\mathbb{Z}_p} x_{(n-j)} d\mu_1(x).$$

Combining the above equations with (2.20) and (2.21), we get

$$(7.10) \quad \int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) D_{n-j}$$

and

$$(7.11) \quad \int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = 2 \sum_{j=0}^n \sum_{l=0}^{n-j} (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j} S_1(n-j, l) B_l}{(2\lambda - 2)^{j+1}}.$$

Combining (7.10) with (7.11), we arrive at the desired result. \square

8. New Sequences Containing Bernoulli Numbers and Euler Numbers

In this section, we examine p -adic integrals of the function

$$J(x) = x_{(n)}x^{(m)}.$$

Moreover, we give some applications of these integrals. With the help of the integrals of this special function $J(x)$, we define two new sequences containing the Bernoulli numbers of the first kind and the Euler numbers of the first kind, respectively. We give some properties of these two sequences. We also prove that the general term of these sequences can be written in terms of the central factorial numbers. We also give some identities and relations involving the Bernoulli numbers of the first kind, the Euler numbers of the first kind, the Stirling numbers, the Lah numbers, and the central factorial numbers.

Let's start this section with the following questions:

How can we compute the following integrals:

Question 1:

$$\int_{\mathbb{Z}_p} J(x) d\mu_1(x) = ?$$

Question 2:

$$\int_{\mathbb{Z}_p} J(x) d\mu_{-1}(x) = ?$$

By using (1.55), we have the following identity:

$$(8.1) \quad x_{(n)}x^{(m)} = \sum_{k=1}^m |L(m, k)| x_{(k)}x_{(n)}.$$

By applying the Volkenborn integral to (8.1), we get

$$\int_{\mathbb{Z}_p} x_{(n)}x^{(m)} d\mu_1(x) = \sum_{k=1}^m |L(m, k)| \int_{\mathbb{Z}_p} x_{(k)}x_{(n)} d\mu_1(x).$$

Combining the above equation with (3.24), we arrive at the following theorem. The result of the following theorem gives us the solution of the Question 1.

Theorem 8.1. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)}x^{(m)} d\mu_1(x) = \sum_{k=1}^m \sum_{j=0}^n (-1)^{k+n-j} \binom{n}{j} \binom{k}{j} \frac{j!(n+k-j)! |L(m, k)|}{n+k-j+1}.$$

By applying the fermionic p -adic integral to (8.1), we get

$$\int_{\mathbb{Z}_p} x_{(n)} x^{(m)} d\mu_{-1}(x) = \sum_{k=1}^m |L(m, k)| \int_{\mathbb{Z}_p} x_{(k)} x_{(n)} d\mu_{-1}(x).$$

Combining the above equation with (6.4), we arrive at the following theorem. The result of the following theorem gives us the solution of the Question 2.

Theorem 8.2. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} x^{(m)} d\mu_{-1}(x) = \sum_{k=1}^m \sum_{j=0}^n (-1)^{n+k-j} \binom{n}{j} \binom{k}{j} \frac{j!(n+k-j)! |L(m, k)|}{2^{n+k-j}}.$$

Substituting $m = n$ into Question 1 and Question 2, we define the following sequences containing the Bernoulli numbers of the first kind and the Euler numbers of the first kind, respectively:

$$\begin{aligned} (8.2) \quad \mathcal{Y}(n, B) &= \int_{\mathbb{Z}_p} x_{(n)} x^{(n)} d\mu_1(x) \\ &= \int_{\mathbb{Z}_p} x^2(x^2-1)(x-2^2)(x^2-3^2)\cdots(x^2-(n-1)^2) d\mu_1(x) \end{aligned}$$

and

$$\begin{aligned} (8.3) \quad \mathcal{Y}(n, E) &= \int_{\mathbb{Z}_p} x_{(n)} x^{(n)} d\mu_{-1}(x) \\ &= \int_{\mathbb{Z}_p} x^2(x^2-1)(x-2^2)(x^2-3^2)\cdots(x^2-(n-1)^2) d\mu_{-1}(x). \end{aligned}$$

By using (2.15) and (3.32), we compute few values of the sequences given by (8.2) and (8.3), respectively, as follows:

$$\begin{aligned} \mathcal{Y}(0, B) &= B_0, \\ \mathcal{Y}(1, B) &= B_2, \\ \mathcal{Y}(2, B) &= B_4 - B_2, \\ \mathcal{Y}(3, B) &= B_6 - 5B_4 + 4B_2, \\ \mathcal{Y}(4, B) &= B_8 - 14B_6 + 49B_4 - 36B_2, \\ \mathcal{Y}(5, B) &= B_{10} - 30B_8 + 273B_6 - 870B_4 + 576B_2 \\ \mathcal{Y}(6, B) &= B_{12} - 55B_{10} + 1023B_8 - 7645B_6 + 21076B_4 - 14400B_2, \dots \end{aligned}$$

and

$$\begin{aligned}
\mathcal{Y}(0, E) &= E_0, \\
\mathcal{Y}(1, E) &= E_2, \\
\mathcal{Y}(2, E) &= E_4 - E_2, \\
\mathcal{Y}(3, E) &= E_6 - 5E_4 + 4E_2, \\
\mathcal{Y}(4, E) &= E_8 - 14E_6 + 49E_4 - 36E_2, \\
\mathcal{Y}(5, E) &= E_{10} - 30E_8 + 273E_6 - 870E_4 + 576E_2 \\
\mathcal{Y}(6, E) &= E_{12} - 55E_{10} + 1023E_8 - 7645E_6 + 21076E_4 - 14400E_2, \dots
\end{aligned}$$

When the integrals, given by (8.2) and (8.3), are calculated for the special values of the number n , we can observe that the row numbers given in the matrix representation of the central factorial numbers $t(i, j)$ in the equation (1.59) and the coefficients of the Bernoulli numbers of the first kind and the Euler numbers of the first kind are equal. Therefore, we arrive at the following theorems:

Theorem 8.3. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{Y}(n, B) = \sum_{k=1}^n t(2n, 2k) B_{2k}.$$

Proof. By applying the Volkenborn integral to the following well-known equation

$$(8.4) \quad x^2(x^2 - 1)(x - 2^2)(x^2 - 3^2) \cdots (x^2 - (n - 1)^2) = \sum_{k=1}^n t(2n, 2k) x^{2k}$$

(cf. [24, p. 430], [64]), we get

$$\int_{\mathbb{Z}_p} x^2(x^2 - 1)(x - 2^2)(x^2 - 3^2) \cdots (x^2 - (n - 1)^2) d\mu_1(x) = \sum_{k=1}^n t(2n, 2k) \int_{\mathbb{Z}_p} x^{2k} d\mu_1(x).$$

Combining the above equation with (2.15), we arrive at the desired result. \square

Theorem 8.4. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned}
\mathcal{Y}(n, E) &= \sum_{k=1}^{2n} t(2n, 2k) E_{2k} \\
&= 0.
\end{aligned}$$

Proof. By applying the fermionic p -integral to the equation (8.4), we get

$$\int_{\mathbb{Z}_p} x^2(x^2 - 1)(x^2 - 2^2)(x^2 - 3^2) \cdots (x^2 - (n - 1)^2) d\mu_{-1}(x) = \sum_{k=1}^n t(2n, 2k) \int_{\mathbb{Z}_p} x^{2k} d\mu_{-1}(x),$$

(cf. [24, p. 430], [64]). Combining the above equation with (3.32), we arrive at the desired result. \square

Remark 8.5. In [113], we defined two other kinds of sequences including the Bernoulli numbers and polynomials of the first kind, and the Euler numbers and polynomials of the first kind. Let's briefly give information about two of them: The sequence $(Y_1(n : B))$ is associated with the Bernoulli numbers of the first kind. That is, $Y_1(0 : B) = B_0 = 1$, $Y_1(1 : B) = B_1 = -\frac{1}{2}$, $Y_1(2 : B) = B_2 - B_1$, $Y_1(3 : B) = B_3 - 3B_2 + 2B_1, \dots$. If we continue to calculate the terms of the sequence $(Y_1(n : B))$ in this way, the general term of this sequence is given by the following formula including the Daehee numbers:

$$(8.5) \quad Y_1(n : B) = D_n.$$

The sequence $(y_2(n : E))$ is associated with the Euler numbers of the first kind. That is, $y_1(0 : E) = y_2(0 : E) = E_0 = 1$ and $y_1(1 : E) = y_2(1 : E) = E_1 = -\frac{1}{2}$, $y_1(2 : E) = E_2 - E_1$, $y_1(3 : E) = E_3 - 3E_2 + 2E_1, \dots$. Similarly, if we continue to calculate the terms of the sequence $(y_2(n : E))$ in this way, the general term of this sequence is given by the following formula including the Changhee numbers:

$$y_1(n : E) = Ch_n.$$

In this paper, we do not consider whether there is any relationship between the sequences $Y_1(n : B)$ and the sequence $y_1(n : E)$ and the newly defined the sequence $\mathcal{Y}(n, B)$ and the sequence $\mathcal{Y}(n, E)$. Perhaps the sequence $\mathcal{Y}(n, B)$ and the sequence $\mathcal{Y}(n, E)$ may be subsequences of the sequences $Y_1(n : B)$ and the sequence $y_1(n : E)$, respectively.

The following theorem gives us that Bernoulli numbers of the first kind can be computed with the help of the central factorial numbers of the second kind $T(n, k)$ and the sequence $\mathcal{Y}(k, B)$.

Theorem 8.6. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_{2n} = \sum_{k=0}^n T(n, k) \mathcal{Y}(k, B).$$

Proof. By applying the Volkenborn integral to the following well-known equation

$$(8.6) \quad x^n = \sum_{k=0}^n T(n, k) x(x-1)(x-2^2)(x-3^2) \cdots (x-(n-1)^2),$$

(cf. [24, p. 430], [64]). By replacing x by x^2 into (8.6), we have

$$x^{2n} = \sum_{k=0}^n T(n, k) x^2(x^2-1)(x^2-2^2)(x^2-3^2) \cdots (x^2-(n-1)^2).$$

By applying the Volkenborn integral to the above equation, we get

$$\int_{\mathbb{Z}_p} x^{2n} d\mu_1(x) = \sum_{k=0}^n T(n, k) \int_{\mathbb{Z}_p} x^2(x^2-1)(x^2-2^2)(x^2-3^2) \cdots (x^2-(n-1)^2) d\mu_1(x).$$

Combining the above equation with (2.15) and (8.2), we arrive at the desired result. \square

The following theorem shows that the Euler numbers of the first kind can be computed with the help of the central factorial numbers of the second kind $T(n, k)$ and the sequence $\mathcal{Y}(k, E)$.

Theorem 8.7. *Let $n \in \mathbb{N}_0$. Then we have*

$$E_{2n} = \sum_{k=0}^n T(n, k) \mathcal{Y}(k, E).$$

Proof. By applying the Volkenborn integral to the following well-known equation

$$(8.7) \quad x^n = \sum_{k=0}^n T(n, k) x(x-1)(x-2^2)(x-3^2) \cdots (x-(n-1)^2).$$

By replacing x by x^2 into (8.7), we have

$$x^{2n} = \sum_{k=0}^n T(n, k) x^2(x^2-1)(x^2-2^2)(x^2-3^2) \cdots (x^2-(n-1)^2).$$

By applying the fermionic p -adic integral to the above equation, we get

$$\int_{\mathbb{Z}_p} x^{2n} d\mu_{-1}(x) = \sum_{k=0}^n T(n, k) \int_{\mathbb{Z}_p} x^2(x^2-1)(x^2-2^2)(x^2-3^2) \cdots (x^2-(n-1)^2) d\mu_{-1}(x).$$

Combining the above equation with (3.32) and (8.3), we arrive at the desired result. \square

We now give another solutions of Question 1 and Question 2 in theorems stated below.

Combining (8.2) with (1.34) and (1.55), we get

$$\mathcal{Y}(n, B) = \sum_{j=0}^n \sum_{k=1}^n S_1(n, j) |L(n, k)| \int_{\mathbb{Z}_p} x^j x_{(k)} d\mu_1(x).$$

Combining the above equation with (3.22), we arrive at the following theorem:

Theorem 8.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{Y}(n, B) = \sum_{j=0}^n \sum_{k=1}^n \sum_{m=0}^k S_1(n, j) S_1(k, m) B_{j+m} |L(n, k)|.$$

Combining (8.3) with (1.34) and (1.55), we get

$$\mathcal{Y}(n, E) = \sum_{j=0}^n \sum_{k=1}^n S_1(n, j) |L(n, k)| \int_{\mathbb{Z}_p} x^j x_{(k)} d\mu_{-1}(x).$$

Combining the above equation with (6.1), we arrive at the following theorem:

Theorem 8.9. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{Y}(n, E) = \sum_{j=0}^n \sum_{k=1}^n \sum_{m=0}^k S_1(n, j) S_1(k, m) E_{j+m} |L(n, k)|.$$

9. Conclusion

One of the main objectives of this paper is to gather p -adic integral formulas together and to provide convenience to readers. The p -adic integrals formulas given in this article are divided into two classes. The first is the bosonic p -adic deformed or the Volkenborn integral formulas and the other is the fermionic p -adic integral formulas.

The methods used in this article can be determined as p -adic integral technique and its integral equations, generating functions and other special function techniques.

With the help of the deformed p -adic (the Volkenborn integral), many different generating functions can be constructed. These generating functions are including several different families of special numbers, families of polynomials and families of special functions. Especially, by using the Volkenborn integral, generating functions for Bernoulli type numbers and polynomials have been constructed. On the other hand, by using the fermionic p -adic integral, generating functions for Euler type numbers and polynomials and Genocchi type numbers and polynomials have been constructed. Besides, generating functions are one of the most important areas of the most used mathematics of the last century. Because, with the help of these functions, they are used in the study of the fundamental properties of families of special numbers and polynomials, finding some special relations and formulas for these families of special numbers and polynomials, in the solution of differential equations, in mathematical modeling theory involving real world problems and in other fields.

In this article we give and classify not only new formulas of p -adic integrals, but also previously well known p -adic integrals formulas. These p -adic integral formulas including several important well-known families of special numbers and special polynomials such as the Bernoulli numbers and polynomials of the first kind, the Euler numbers and polynomials of the first kind, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the Central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums, and the Dedekind sums. We give two new sequences including the Bernoulli numbers of the first kind, the Euler numbers of the first kind, and the central factorial numbers. We give various different computation formulas and identities for these sequences.

The results of this may contribute to p -adic analysis especially ultrametric calculus, and also some areas of physics and mathematical physics including combinational physics, q -quantum mechanic, q -quantum model. These results will also shed light on the fields of researchers working in not only theoretical sciences, but also applied sciences.

10. Appendix: Tables of p -adic integral formulas

In this section, we finalize this paper along with an appendix of the following tables, which contains numerous p -adic integral formulas, that may constitute very useful source to researchers working in the related areas. These tables are given below include both new and previously known formulas for the Volkenborn integral and the fermionic p -adic integral.

Table 1. Volkenborn Integral Formulas

Volkenborn Integral Formulas	
Including binomial coefficients	Including falling or raising factorials
$\int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = (-1)^n \binom{1}{n+1} \left(\frac{1}{n+1} - \frac{1}{n} \right).$	$\int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x) = \sum_{l=0}^n S_1(n, l) B_l = D_n = \frac{(-1)^n n!}{n+1}.$
$\int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_1(x) = \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} \frac{1}{m+1}.$	$\int_{\mathbb{Z}_p} (x+1)_{(n+1)} d\mu_1(x) = \frac{(-1)^n}{n+2} n!.$
$\int_{\mathbb{Z}_p} \binom{x+m}{n} d\mu_1(x) = \sum_{k=0}^n (-1)^k \binom{m}{n-k} \frac{1}{k+1}.$	$\int_{\mathbb{Z}_p} x_{(m)} (x-m)_{(n)} d\mu_1(x) = \sum_{k=0}^{n+m} S_1(m+n, k) B_k = D_{m+n}$ $= (-1)^{m+n} \frac{(m+n)!}{m+n+1}.$
$\int_{\mathbb{Z}_p} \binom{x-2}{n-1} d\mu_1(x) = (-1)^{-n} \sum_{k=1}^n \frac{k}{k+1}.$	$\int_{\mathbb{Z}_p} \frac{x_{(n+1)}}{x} d\mu_1(x) = \sum_{k=0}^n (-1)^n n_{(n-k)} \frac{k!}{k+1}.$
$\int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_1(x) = H_n$	$\int_{\mathbb{Z}_p} x x^{(n)} d\mu_1(x) = \sum_{k=1}^n (-1)^{k+1} \binom{n-1}{k-1} \frac{n!}{k^2+3k+2} = \sum_{k=1}^n C(n, k) B_{k+1}.$
$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_1(x) = \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \sum_{l=0}^k \frac{S_1(k, l) B_{v+l}}{k!}.$	$\int_{\mathbb{Z}_p} x x_{(n)} d\mu_1(x) = \sum_{k=1}^n S_1(n, k-1) B_k + B_{n+1} = (-1)^{n+1} \frac{n!}{n^2+3n+2}.$
	$\int_{\mathbb{Z}_p} x_{(n)} x^{(m)} d\mu_1(x) = \sum_{k=1}^m \sum_{j=0}^n (-1)^{k+n-j} \binom{n}{j} \binom{k}{j} \frac{j!(n+k-j)! L(m, k) }{n+k-j+1}.$

Table 2. Volkenborn Integral Formulas

Volkenborn Integral formulas		
Including exponential functions	Including trigonometric functions	Including power functions
$\int_{\mathbb{Z}_p} a^x d\mu_1(x) = \frac{1}{a-1} \log_p(a).$	$\int_{\mathbb{Z}_p} \cos(ax) d\mu_1(x) = \frac{a \sin(a)}{2(1 - \cos(a))}$ $= \frac{a}{2} \cot\left(\frac{a}{2}\right).$	$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n.$
$\int_{\mathbb{Z}_p} e^{ax} d\mu_1(x) = \frac{a}{e^a - 1}.$	$\int_{\mathbb{Z}_p} \sin(ax) d\mu_1(x) = -\frac{a}{2}.$	$\int_{j+p^n\mathbb{Z}_p} x^m d\mu_1(x) = p^{n(m-1)} B_m\left(\frac{j}{p^n}\right)$
$\int_{\mathbb{Z}_p} (1+t)^x d\mu_1(x) = \frac{1}{t} \log(1+t).$	$\int_{\mathbb{Z}_p} \tan(ax) d\mu_1(x) = -\frac{a}{2}.$	$\int_{\mathbb{Z}_p} (z+x)^n d\mu_1(x) = B_n(z)$
$\int_{\mathbb{Z}_p} \lambda^x e^{t(x+v)} d\mu_1(x) = \frac{\log \lambda + t}{\lambda e^t - 1} e^{tv}.$	$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_1(x) = -\frac{a}{2}.$	$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_1(x) = \sum_{k=0}^{2n} t(2n, k) B_{2k}.$
$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = B_n(\lambda)$ $= \frac{n H_{n-1}(\lambda^{-1})}{\lambda - 1}.$		$\int_{\mathbb{Z}_p} x \prod_{k=1}^n \left(x^2 - \frac{(2k-1)^2}{4} \right) d\mu_1(x)$ $= -\frac{1}{2} \frac{d}{dx} \left\{ x^{[2n+1]} \right\} \Big _{x=0}$

Table 3. Further Volkenborn Integral Formulas Including Special Functions

Further Volkenborn Integral Formulas Including Special Functions
$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_1(x) = \sum_{k=0}^n t(n, k) B_k + \left(\frac{n-2}{2}\right)^{2n-2} \sum_{k=0}^{n-2} t(n-2, k) B_k$
$\int_{\mathbb{Z}_p} x^{[n]} d\mu_1(x) = \sum_{k=0}^n t(n, k) B_k$
$\int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_1(x) = \sum_{v=0}^n \binom{n}{v} s_v(\lambda, \mu) \sum_{l=0}^{n-v} S_1(n-v, l) B_l$
$\sum_{k=0}^n (-1)^{k-n} \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) = \sum_{j=0}^n \binom{n}{j} (-2)^{n-j} B_{n-j}.$
$\sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-1)^{n+m-j} 2^{n-j} S_2(n-j, m) \frac{m!}{m+1}.$
$\sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-2)^{n-j} S_2(n-j, m) D_m.$

Table 4. Fermionic p -adic Integral Formulas

Fermionic p -adic Integral Formulas	
Including binomial coefficients	Including power or exponential functions
$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}(x) = (-1)^n 2^{-n}.$	$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x).$
$\int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_{-1}(x) = \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} 2^{-m}.$	$E_n(z) = \int_{\mathbb{Z}_p} (z+x)^n d\mu_{-1}(x).$
$\int_{\mathbb{Z}_p} \left\{ x \binom{x-2}{n-1} + x(x-1) \binom{n-3}{n-2} \right\} d\mu_{-1}(x) = (-1)^n \sum_{k=0}^n \frac{k^2}{2^k}.$	$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \sum_{j=0}^k \sum_{l=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \frac{S_1(k, l) E_{0+l}}{k!}.$
$\int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_{-1}(x) = \sum_{k=0}^n \frac{(-1)^k}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j+n}{n}$ $= \sum_{k=0}^n E_k \sum_{j=0}^n \binom{n}{j} \frac{S_1(j, k)}{j!}.$	$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_{-1}(x) = \sum_{k=0}^{2n} t(2n, k) E_{2k}.$
$\int_{\mathbb{Z}_p} \binom{mx}{n} d\mu_{-1}(x) = \sum_{k=0}^n \frac{(-1)^k}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{mk-nj}{n}.$	$\int_{\mathbb{Z}_p} \lambda^x x^v d\mu_{-1}(x) = \mathcal{E}_n(\lambda).$
$\int_{\mathbb{Z}_p} \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r.$	$\int_{\mathbb{Z}_p} \lambda^x e^{tx} d\mu_{-1}(x) = \frac{2}{\lambda e^t + 1}.$
$\int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_{-1}(x) = \sum_{k=1}^n 2^{-k}.$	
$\int_{\mathbb{Z}_p} \binom{x+n+\frac{1}{2}}{n} d\mu_{-1}(x) = (2n+1) \binom{2n}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{2^{k-2n}}{(2k+1) \binom{2k}{k}}.$	

Table 5. Fermionic p -adic Integral Formulas Including Falling or Raising Factorials

Fermionic p-adic Integral Formulas Including Falling or Raising Factorials	
$\int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x) = (-1)^n 2^{-n} n!$	$\int_{\mathbb{Z}_p} (x+1)_{(n)} d\mu_{-1}(x) = (-1)^{n+1} \frac{1}{2^n} n!$
$\int_{\mathbb{Z}_p} \frac{x^{(n+1)}}{x} d\mu_{-1}(x) = \sum_{k=0}^n (-1)^n n_{(n-k)} \frac{k!}{2^k}$	$\int_{\mathbb{Z}_p} x x_{(n)} d\mu_{-1}(x) = (-1)^n \frac{(n-1)}{2^{n+1}} n!$
$\int_{\mathbb{Z}_p} x x^{(n)} d\mu_{-1}(x) = \sum_{k=1}^n (-1)^k \binom{n-1}{k-1} \frac{(k-1)}{2^{k+1}} n!$	$\int_{\mathbb{Z}_p} x^m x_{(n)} d\mu_{-1}(x) = \sum_{k=0}^n S_1(n, k) E_{k+m}$
$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_{-1}(x) = \sum_{k=0}^m (-1)^{m+n-k} \binom{m}{k} \binom{n}{k} \frac{k!(m+n-k)!}{2^{m+n-k}}$	
$\int_{\mathbb{Z}_p} x_{(n)} x^{(m)} d\mu_{-1}(x) = \sum_{k=1}^m \sum_{j=0}^n (-1)^{n+k-j} \binom{n}{j} \binom{k}{j} \frac{j!(n+k-j)! L(m, k) }{2^{n+k-j}}$	
$\int_{\mathbb{Z}_p} x_{(m)} (x-m)_{(n)} d\mu_{-1}(x) = \sum_{k=0}^{n+m} S_1(n+m, k) E_k$ $= Ch_{m+n}$ $= (-1)^{m+n} \frac{(m+n)!}{2^{m+n}}$	
$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_{-1}(x) d\mu_{-1}(y) = \sum_{l,m=1}^k (-1)^{l+m} 2^{-m-l} l! m! C_{l,m}^{(k)}$ $= \sum_{m=0}^k S_1(k, m) (E_m)^2$	

Table 6. Fermionic p -adic Integral Formulas Including Trigonometric Functions

Fermionic p-adic Integral Formulas Including Trigonometric Functions
$\int_{\mathbb{Z}_p} \cos(ax) d\mu_{-1}(x) = 1.$
$\int_{\mathbb{Z}_p} \sin(ax) d\mu_{-1}(x) = -\frac{\sin(a)}{\cos(a) + 1}.$
$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_{-1}(x) = 1.$

Table 7. Further Fermionic p -adic Integral Formulas Including Special Functions

Further Fermionic p-adic Integral Formulas Including Special Functions
$\int_{\mathbb{Z}_p} x^{[n]} d\mu_{-1}(x) = \sum_{k=0}^n t(n, k) E_k.$
$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_{-1}(x) = \sum_{k=0}^n t(n, k) E_k + \left(\frac{n-2}{2}\right)^2 \sum_{k=0}^{n-2} t(n-2, k) E_k.$
$\begin{aligned} \int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_{-1}(x) &= \sum_{v=0}^n \binom{n}{v} s_v(\lambda, \mu) Ch_{n-v} \\ &= \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} \frac{s_v(\lambda, \mu) (n-v)!}{2^{n-v}}. \end{aligned}$

Acknowledgments

The author would like to thank to the referees for their high-quality and valuable reports.

The author was supported by the Scientific Research Project Administration of Akdeniz University.

REFERENCES

- [1] M. Acikgoz, S. Araci, *On the Generating Function for Bernstein Polynomials*, Amer. Institute of Physics Conference Proceedings **CP1281**, 1141-1144, 2010.
- [2] M. Aigner, *A Course in Enumeration*, Springer-Verlag, Berlin-Heidelberg, 2007.
- [3] Y. Amice, *Mesures p -adiques*, Séminaire Delange-Pisot-Poitou, Théorie des nombres **6 (2)**, 1-6, 1964-1965.
- [4] Y. Amice, *Integration p -adique Selon A. Volkenborn*, Séminaire Delange-Pisot-Poitou, Théorie des nombres **13 (2)**, G4, G1-G9, 1971-1972.
- [5] S. Araci, M. Acikgoz, E. Sen, *On the extended Kim's p -adic q -deformed fermionic integrals in the p -adic integer ring*, J. Number Theory **133 (10)**, 3348-3361, 2013.
- [6] S. Araci, U. Duran, M. Acikgoz, *(p, q) -Volkenborn integration*, J. Number Theory **171**, 18-30, 2017.
- [7] T. M. Apostol, *On the Lerch Zeta Function*, Pacific J. Math. **1 (2)**, 161-167, 1951.
- [8] A. Bayad, Y. Simsek, H. M. Srivastava, *Some Array Type Polynomials Associated with Special Numbers and Polynomials*, Appl. Math. Compute. **244**, 149-157, 2014.
- [9] H. Belbachir, I. E. Bousbaa, *Associated Lah Numbers and r -Stirling Numbers*, arXiv:1404.5573v2 (math.CO), 12 May 2014.
- [10] K. N. Boyadzhiev, *Close Encounters with the Stirling Numbers of the Second Kind*, Math. Mag. **85**, 252-266, 2012.
- [11] K. N. Boyadzhiev, *Binomial Transform and the Backward Difference*, <https://arxiv.org/vc/arxiv/papers/1410/1410.3014v2.pdf>.
- [12] L. Carlitz, *The Reciprocity Theorem for Dedekind Sums*, Pacific J. Math. **3**, 523-527, 1953.
- [13] C. H. Chang, C. W. Ha, *A Multiplication Theorem for the Lerch Zeta Function and Explicit Representations of the Bernoulli and Euler Polynomials*, J. Math. Anal. Appl. **315**, 758-767, 2006.
- [14] C. A. Charalambides, *Combinatorial Methods in Discrete Distributions*, A John Wiley and Sons, Inc., Publication, 2015.
- [15] C. A. Charalambides, *Enumerative Combinatorics*, Chapman and Hall/Crc, Press Company, London, New York, 2002.
- [16] J. Cigler, *Fibonacci Polynomials and Central Factorial Numbers*, preprint.
- [17] J. Choi, H.M. Srivastava, *Some Summation Formulas Involving Harmonic Numbers and Generalized Harmonic Numbers*, Mathematical and Computer Modelling **54**, 2220-2234, 2011.
- [18] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Reidel, Dordrecht and Boston, 1974.
- [19] H. Coskun, *Multiple Bracket Function, Stirling Number, and Lah Number Identities*, arXiv:1212.6573v2 (math.NT) 17 Jun 2015.
- [20] G. E. W. Dickson, L.E., H.H. Mitchell, H.S. Vandiver, *Algebraic Numbers. Report of the Committee on Algebraic Numbers, National Research Council, Bulletin of the National Research Council Vol. 5, Part 3. Number 28*, National Academy of Sciences, Washington, 1923.
- [21] G. B. Djordjevic, G. V. Milovanovic, *Special Classes of Polynomials*, University of Nis, Faculty of Technology Leskovac, 2014.

- [22] K. N. Boyadzhiev, *Lah numbers, Laguerre Polynomials of Order Negative One, and the n th Derivative of $\exp(\frac{1}{x})$* , Acta Univ. Sapientiae, Mathematica **8** (1), 22-31, 2016.
- [23] R. A. Brigham II, *A Harmonic M -Factorial Function and Applications*, Doctoral Dissertations, 2557, 2017, https://scholarsmine.mst.edu/doctoral_dissertations/2557.
- [24] P. L. Butzer, K. Schmidt, E. L. Stark, L. Vogt, *Central Factorial Numbers; Their Main Properties and Some Applications*, Numer. Funct. Anal. and Optimiz. **10** (5-6), 419-488, 1989.
- [25] P. F. Byrd, *New Relations Between Fibonacci and Bernoulli Numbers*, Fibonacci Quarterly **13**, 111-114, 1975.
- [26] N. P. Cakic, G. V. Milovanovic, *On Generalized Stirling Numbers and Polynomials*, Mathematica Balkanica **18**, 241-248, 2004.
- [27] Y. Do, D. Lim, *On (h, q) -Daehee Numbers and Polynomials*, Adv. Difference Equ. **107**, 1-9, 2015.
- [28] B. S. El-Desouky, A. Mustafa, *New Results and Matrix Representation for Daehee and Bernoulli Numbers and Polynomials*, Applied Mathematical Sciences **9** (73), 3593-3610, 2015, arXiv:1412.8259v1 (math.CO) 29 Dec 2014.
- [29] B. S. El-Desouky, R. S. Goma, *Multiparameter Poly-Cauchy and Poly-Bernoulli Numbers and Polynomials*, International J. Mathematical Analysis **9** (53), 2619-2633, 2015, arXiv:1410.5300v1 (math.CO) 20 Oct 2014.
- [30] A. Garsia, J. Remmel, *A Combinatorial Interpretation of q -Derangement and q -Laguerre Numbers*, European J. Combin. **1**, 47-59, 1980.
- [31] R. Golombek, *Aufgabe 1088*, El. Math. **49**, 126-127, 1994.
- [32] I. J. Good, *The Number of Ordering of n Candidates When Ties are Permitted*, Fibonacci Quart. **13**, 11-18, 1975.
- [33] H. W. Gould, *Fundamentals of Series*, <https://math.wvu.edu/~hgould/Vol.3.PDF>.
- [34] H. W. Gould, *Combinatorial Numbers and Associated Identities*, <https://math.wvu.edu/~hgould/Vol.7.PDF>.
- [35] H.W. Gould, *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, Revised ed., Morgantown Printing and Binding Company, Morgantown, West Virginia, 1972.
- [36] B. N. Guo, F. Qi, *An Explicit Formula for Bernoulli Numbers in Terms of Stirling Numbers of the Second Kind*, J. Ana. Num. Theor. **3** (1), 27-30, 2015.
- [37] K. Iwasawa, *Lectures on p -adic L -function*, Princeton Univ. Press, Princeton, 1972.
- [38] L.-C. Jang, W. Kim, H.-I. Kwon, *On degenerate Daehee polynomials and numbers of the third kind*, J. Computational Appl. Math., 2019, <https://doi.org/10.1016/j.cam.2019.112343>.
- [39] L. C. Jang, T. Kim, *A New Approach to q -Euler Numbers and Polynomials*, J. Concr. Appl. Math. **6**, 159-168, 2008.
- [40] L. C. Jang, T. Kim, D. H. Lee, D. W. Park, *An Application of Polylogarithms in the Analogs of Genocchi Numbers*, Notes Number Theory Discrete Math. **7** (3), 65-69, 2001.
- [41] L. C. Jang, H. K. Pak, *Non-archimedean Integration Associated with q -Bernoulli Numbers*, Proc. Jangjeon Math. Soc. **5** (2), 125-129, 2002.
- [42] H. Jolany, H. Sharifi, R. E. Alikelaye, *Some Results for the Apostol-Genocchi Polynomials of Higher Order*, Bull. Malays. Math. Sci. Soc. **36** (2), 465-479, 2013.
- [43] J. Jeong, D.-J. Kang, S.-H. Rim, *Symmetry Identities of Changhee Polynomials of Type Two*, Symmetry **10**, 740, 2018.
- [44] J.-W. Park, G.-W. Jang, J. Kwon, *The λ -analogue degenerate Changhee polynomials and numbers*, Global J. Pure Appl. Math. ISSN 0973-1768 **13** (3), 893-900, 2017.
- [45] C. Jordan, *Calculus of Finite Differences (2nd ed.)*, Chelsea Publishing Company, New York, 1950.

- [46] N. Kilar, Y. Simsek, *A New Family of Fubini Type Numbers and Polynomials Associated with Apostol-Bernoulli Numbers and Polynomials*, J. Korean Math. Soc. **54** (5), 1605-1621, 2017.
- [47] D. S. Kim, D. V. Dolgy, D. Kim, T. Kim, *Some Identities on r -central Factorial Numbers and r -central Bell Polynomials*, Adv. Difference Equ. **245**, 1-11, 2019.
- [48] D. S. Kim, T. Kim, *Some New Identities of Frobenius-Euler Numbers and Polynomials*, J. Ineq. Appl. **307**, 1-10, 2012.
- [49] D. S. Kim, T. Kim, *Daehee Numbers and Polynomials*, Appl. Math. Sci. (Ruse) **7** (120), 5969-5976, 2013.
- [50] D. S. Kim, T. Kim, *A Note on Boole Polynomials*, Integral Transforms Spec. Funct. **25** (8), 627-633, 2014.
- [51] D. S. Kim, T. Kim, *Some Identities of Degenerate Special Polynomials*, Open Math. **13**, 380-389, 2015.
- [52] D. S. Kim, T. Kim, J. Seo, *A note on Changhee numbers and polynomials*, Adv. Stud. Theor. Phys. **7**, 993-1003, 2013.
- [53] D. S. Kim, T. Kim, J. J. Seo, T. Komatsu, *Barnes' Multiple Frobenius-Euler and Poly-Bernoulli Mixed-type Polynomials*, Adv. Difference Equ. **92**, 1-16, 2014.
- [54] T. Kim, *An Analogue of Bernoulli Numbers and Their Congruences*, Rep. Fac. Sci. Engrg. Saga Univ. Math. **22** (2), 21-26, 1994.
- [55] T. Kim, *The Modified q -Euler Numbers and Polynomials*, arXiv:math/0702523v1 (math.NT) 18 Feb 2007.
- [56] T. Kim, *On a q -analogue of the p -adic log gamma functions*, J. Number Theory **76**, 320-329, 1999.
- [57] T. Kim, *q -Volkenborn Integration*, Russ. J. Math. Phys. **19**, 288-299, 2002.
- [58] T. Kim, *An Invariant p -adic Integral Associated with Daehee Numbers*, Integral Transforms Spec. Funct. **13** (1), 65-69, 2002.
- [59] T. Kim, *q -Euler Numbers and Polynomials Associated with p -adic q -Integral and Basic q -zeta Function*, Trend Math. Information Center Math. Sciences **9**, 7-12, 2006.
- [60] T. Kim, *On the Analogs of Euler Numbers and Polynomials Associated with p -adic q -integral on \mathbb{Z}_p at $q = 1$* , J. Math. Anal. Appl. **331** (2), 779-792, 2007.
- [61] T. Kim, *An Invariant p -adic q -integral on \mathbb{Z}_p* , Appl. Math. Letters **21**, 105-108, 2008.
- [62] T. Kim, *p -adic l -functions and Sums of Powers*, arXiv:math/0605703v1 (math.NT) 27 May 2006.
- [63] T. Kim, *On the q -extension of Euler and Genocchi Numbers*, J. Math. Anal. Appl. **326** (2), 1458-1465, 2007.
- [64] T. Kim, *A Note on Central Factorial Numbers*, Proceed. Jangjeon Math. Soc. **21** (4), 575-588, 2018.
- [65] D. Kim, H. O. Ayna, Y. Simsek, A. Yardimci, *New Families of Special Numbers and Polynomials Arising from Applications of p -adic q -integrals*, Adv. Difference Equ. **207**, 1-11, 2017.
- [66] T. Kim, J. Choi, Y. H. Kim, C. S. Ryoo, *On the Fermionic p -adic Integral Representation of Bernstein Polynomials Associated with Euler Numbers and Polynomials*, J. Inequal. Appl. **Article ID 864247**, 1-12, 2010.
- [67] T. Kim, D. S. Kim, D. V. Dolgy, J. J. Seo, *Bernoulli Polynomials of the Second Kind and Their Identities Arising from Umbral Calculus*, J. Nonlinear Sci. Appl. **9**, 860-869, 2016.
- [68] T. Kim, D. S. Kim, K. W. Hwang, *Some Identities of Laguerre Polynomials Arising from Differential Equations*, Adv. Differ. Equ. **159**, 1-9, 2016.
- [69] T. Kim, D. S. Kim, G-W. Jang, J. Kwon, *Symmetric Identities for Fubini Polynomials*, Symmetry **10** (6), 219, 2018, <https://doi.org/10.3390/sym10060219>.

- [70] T. Kim, M.S. Kim, L.C. Jang, S.-H. Rim, *New q -Euler Numbers and Polynomials Associated with p -adic q -integrals*, <https://arxiv.org/pdf/0709.0089.pdf>.
- [71] T. Kim, S. H. Rim, *Some q -Bernoulli Numbers of Higher Order Associated with the p -adic q -integrals*, Indian J. Pure Appl. Math. **32** (10), 1565-1570, 2001.
- [72] T. Kim, S.H. Rim, Y. Simsek, D. Kim, *On the Analogs of Bernoulli and Euler Numbers, Related Identities and Zeta and l -functions*, J. Korean Math. Soc. **45** (2), 435-453, 2008.
- [73] M. S. Kim, *On Euler Numbers, Polynomials and Related p -adic Integrals*, J. Number Theory **129** (9), 2166-2179, 2009.
- [74] M. S. Kim, J. W. Son, *Analytic Properties of the q -Volkenborn Integral on the Ring of p -adic Integers*, Bull. Korean Math. Soc. **44** (1), 1-12, 2007.
- [75] M. S. Kim, J. W. Son, *Some Remarks on a q -analogue of Bernoulli Numbers*, J. Korean Math.Soc. **39** (2), 221-236, 2002.
- [76] A. Khrennikov, *p -adic Valued Distributions and Their Applications to the Mathematical Physics*, Kluwer, Dordrecht, 1994.
- [77] I. Kleiner, *A History of Abstract Algebra*, Birkhauser, Boston, 2007.
- [78] N. Koblitz, *p -Adic Numbers, p -adic Analysis, and Zeta-Functions (Second Edition)*, Springer-Verlag, New York, Berlin, Haidellerg, 1977.
- [79] T. Kubota, H.W. Leopoldt, *Eine p -adische theorie der zetawerte I*, J. Reine Angew. Math. **214-215**, 328-339, 1964.
- [80] S. Lang, *Cyclotomic Fields*, Springer Verlag, New York, 1978.
- [81] D. Lim, *On the Twisted Modified q -Daehee Numbers and Polynomials*, Adv. Stud. Theor. Phys. **9** (4), 199-211, 2015.
- [82] G. G. Lorentz, *Bernstein Polynomials*, Chelsea Publishing Company, New York, 1986.
- [83] Q. M. Luo, H. M. Srivastava, *Some Generalizations of the Apostol-Genocchi Polynomials and the Stirling Numbers of the Second Kind*, Appl. Math. Compute. **217**, 5702-5728, 2011.
- [84] D. Merlini, R. Sprugnoli, M. C. Verri, *The Cauchy Numbers*, Discrete Math. **306** (16), 1906-1920, 2006.
- [85] B. Osgood, W. Wu, *Falling Factorials, Generating Functions, and Conjoint Ranking Tables*, J. Integer Seq. **12**, 1-13, 2009.
- [86] H. Ozden, *p -adic q -measure and its applications*, Doctoral dissertation, Uludag University, Bursa, Turkey, 2009.
- [87] H. Ozden, Y. Simsek, *Modification and Unification of the Apostol-type Numbers and Polynomials and Their Applications*, Appl. Math. Compute. **235**, 338-351, 2009.
- [88] H. Ozden, Y. Simsek, I. N. Cangul, *Euler Polynomials Associated with p -adic q -Euler Measure*, Gen. Math. **15** (2-3), 2007.
- [89] J. W. Park, *On a q -analogue of (h, q) -Daehee Numbers and Polynomials of Higher Order*, J. Compute. Anal. Appl. **21** (1), 769-776, 2016.
- [90] A. P. Prudnikov, Yu. A. Bryckov, O. I. Maricev, *Integrals and Series, Vol. 1: Elementary Functions*, Nauka, Moscow, 1981, (in Russian); Translated from the Russian and with a Preface by N.M. Queen, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo and Melbourne, 1986.
- [91] F. Qi, *Explicit Formulas for Computing Bernoulli Numbers of the Second Kind and Stirling Numbers of the First Kind*, Filomat **28** (2), 319-327, 2014.
- [92] F. Qi, X. T. Shi, F. F. Liu, *Several Identities Involving the Falling and Rising Factorials and the Cauchy, Lah, and Stirling Numbers*, Acta Univ. Sapientiae, Mathematica **8** (2), 282-297, 2016.
- [93] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.
- [94] K. F. Riley, M. P. Hobson, S. J. Bence, *Mathematical Methods for Physics and Engineering: A Comprehensive Guide (Third Edition)*, Cambridge University Press, New York, 2006.

- [95] S.-H. Rim, T. Kim, *A Note on p -adic Euler Measure on Z_p* , Russ. J. Math. Phys. **13** (3), 2006.
- [96] S.-H. Rim, T. Kim, S.S. Pyo, *Identities between harmonic, hyperharmonic and Daehee numbers*, J. Inequal Appl. **1**, 168, 2018.
- [97] J. Riordan, *Introduction to Combinatorial Analysis*, Princeton University Press, 1958.
- [98] A. M. Robert, *A Course in p -adic Analysis*, Springer, New York, 2000.
- [99] S. Roman, *The Umbral Calculus*, Dover Publ. Inc., New York, 2005.
- [100] C. S. Ryoo, D. V. Dolgy, H. I. Kwon, Y. S. Jang, *Functional Equations Associated with Generalized Bernoulli Numbers and Polynomials*, Kyungpook Math. J. **55**, 29-39, 2015.
- [101] W. H. Schikhof, *Ultrametric Calculus: An Introduction to p -adic Analysis*, Cambridge Studies in Advanced Mathematics 4, Cambridge University Press, Cambridge, 1984.
- [102] K. Shiratani, S. Yokoyama, *An Application of p -adic Convolutions*, Mem. Fac. Sci. Kyushu Univ. Ser. A Math. **36** (1), 73-83, 1982.
- [103] Y. Simsek, *Twisted p -adic (h, q) - L -functions*, Comput. Math. Appl. **59** (6), 2097-2110, 2010.
- [104] Y. Simsek, *Generating Functions for Generalized Stirling Type Numbers, Array Type Polynomials, Eulerian Type Polynomials and Their Applications*, Fixed Point Theory Appl. **87**, 1-28, 2013.
- [105] Y. Simsek, *Functional Equations from Generating Functions: A Novel Approach to Deriving Identities for the Bernstein Basis Functions*, Fixed Point Theory Appl. **80**, 1-13, 2013.
- [106] Y. Simsek, *Identities Associated with Generalized Stirling Type Numbers and Eulerian Polynomials*, Math. Comput. Appl. **18** (3), 251-263, 2013.
- [107] Y. Simsek, *Special Numbers on Analytic Functions*, Applied Math. **5**, 1091-1098, 2014.
- [108] Y. Simsek, *Analysis of the Bernstein Basis Functions: An Approach to Combinatorial Sums Involving Binomial Coefficients and Catalan Numbers*, Math. Method. Appl. Sci. **38**, 3007-3021, 2015.
- [109] Y. Simsek, *Computation Methods for Combinatorial Sums and Euler Type Numbers Related to New Families of Numbers*, Math. Meth. Appl. Sci. **40** (7), 2347-2361, 2016.
- [110] Y. Simsek, *Apostol Type Daehee Numbers and Polynomials*, Adv. Studies Contemp. Math. **26** (3), 1-12, 2016.
- [111] Y. Simsek, *Analysis of the p -adic q -Volkenborn Integrals: An Approach to Generalized Apostol-type Special Numbers and Polynomials and Their Applications*, Cogent Math. **3**, 1-17, 2016.
- [112] Y. Simsek, *Identities on the Changhee Numbers and Apostol-Daehee Polynomials*, Adv. Stud. Contemp. Math. **27** (2), 199-212, 2017.
- [113] Y. Simsek, *Formulas for p -adic q -integrals Including Falling-Rising Factorials, Combinatorial Sums and Special Numbers*, arXiv:1702.06999v1 (math.NT) 22 Feb 2017.
- [114] Y. Simsek, *New families of Special Numbers for Computing Negative Order Euler Numbers and Related Numbers and Polynomials*, Appl. Anal. Discrete Math. **12**, 1-35, 2018.
- [115] Y. Simsek, *Construction of Some New Families of Apostol-type Numbers and Polynomials via Dirichlet Character and p -adic q -integrals*, Turk. J. Math. **42**, 557-577, 2018.
- [116] Y. Simsek, *Construction method for generating functions of special numbers and polynomials arising from analysis of new operators*, Math. Meth. Appl. Sci. **41**, 6934-6954, 2018.

- [117] Y. Simsek, *Peters Type Polynomials and Numbers and Their Generating Functions: Approach with p -adic Integral Method*, Math Meth Appl Sci., 1-17, 2019, doi:10.1002/mma.5807.
- [118] Y. Simsek, *A New Family of Combinatorial Numbers and Polynomials Associated with Peters Numbers and Polynomials*, to appear in Appl. Anal. Discrete Math.
- [119] Y. Simsek, M. Acikgoz, *A New Generating Function of $(q-)$ Bernstein-type Polynomials and Their Interpolation Function*, Abstr. Appl. Anal. **769095**, 1-12, 2010.
- [120] Y. Simsek, H. M. Srivastava, *A Family of p -adic Twisted Interpolation Functions Associated with the Modified Bernoulli Numbers*, Appl. Math. Comput. **216 (10)**, 2976-2987, 2010.
- [121] Y. Simsek, S.H. Rim, L.-C. Jang, D. J. Kang, J.J. Seo, *A Note on q -Daehee Sums*, Proc. 16th Int. Conf. Jangjeon Math. Soc. **16**, 159-166, 2005.
- [122] H. M. Srivastava, *Some Generalizations and Basic or q -extensions of the Bernoulli, Euler and Genocchi Polynomials*, Appl. Math. Inf. Sci. **5**, 390-444, 2011.
- [123] H. M. Srivastava, J. Choi, *Zeta and q -zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
- [124] H. M. Srivastava, T. Kim, Y. Simsek, *q -Bernoulli Numbers and Polynomials Associated with Multiple q -zeta Functions and Basic L -series*, Russ. J. Math. Phys. **12**, 241-268, 2005.
- [125] H. M. Srivastava, G. D. Liu, *Some Identities and Congruences Involving a Certain Family of Numbers*, Russ. J. Math. Phys. **16**, 536-542, 2009.
- [126] E. Steinitz, *Algebraische Theorie der Krper (2nd ed.)*, Chelsea Publishing Company, Chelsea, 1950.
- [127] B. A. Tangedal, P. T. Young, *On p -adic multiple zeta and log gamma functions*, J. Number Theory **131 (7)**, 1240-1257, 2011.
- [128] N. M. Temme, *Asymptotic Estimates for Laguerre Polynomials*, J. Appl. Math. Physics **41**, 114-126, 1990.
- [129] V. S. Vladimirov, I. V. Volovich, E. I. Zelenov, *p -adic Analysis and Mathematical Physics*, World Scientific, Singapore, 1994.
- [130] A. Volkenborn, *On Generalized p -adic Integration*, Mém. Soc. Math. Fr. **39-40**, 375-384, 1974.
- [131] C.F. Woodcock, *Convolutions on the Ring of p -adic Integers*, J. Lond. Math. Soc. **20 (2)**, 101-108, 1979.
- [132] H. Wang, G. Liu, *An Explicit Formula for Higher Order Bernoulli Polynomials of the Second Kind*, Integers **13**, #A75, 2013.
- [133] G. E. Wahlin, *A New Development of the Theory of Algebraic Numbers*, Transactions of the American Mathematical Society **16 (4)**, 502-508, 1915.
- [134] S. J. Yun, J.W. Park, *On the fully degenerate Daehee numbers and polynomials of the second kind*, Preprints (www.preprints.org) 8 October 2018, doi:10.20944/preprints201810.0129.v1.
- [135] en.wikipedia.org/wiki/Falling_rising_factorials.
- [136] en.wikipedia.org/wiki/Volkenborn_integral.
- [137] en.wikipedia.org/wiki/Lah_number

(Yilmaz Simsek) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE UNIVERSITY OF AKDENIZ TR-07058 ANTALYA-TURKEY
E-mail address: ysimsek@akdeniz.edu.tr