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New Quadratic Transformations of Hypergeometric Functions and Associated Summation Formulas Obtained with the Well-Poised Fractional Calculus Operator

R. Tremblay[a]

[a] *Département d'Informatique et Mathématique, Université du Québec à Chicoutimi, Chicoutimi, Qué., Canada G7H 2B1*

Abstract

Recently, Tremblay and Gaboury used the operator ${}_{g(z)}O^{\alpha}_{\beta}$ and obtained new formulas for the transformation of hypergeometric functions with higher order rational arguments (R.Tremblay and S. Gaboury, Well-poised fractional calculus: obtaining new transformations formulas involving Gauss hypergeometric functions with rational quadratic, cubic and higher degree arguments, Math. Meth. Appl. Sc., (13) (2018), p. 4967-4985). This operator was introduced for the first time in 1974 by Tremblay (R. Tremblay, Une contribution à la théorie de la dérivée fractionnaire [Ph.D. thesis], Laval University, Quebec City, Canada). The main purpose of this article is to illustrate, using numerous examples, the efficiency of this operator in obtaining new results involving special functions. In particular, we deduce twenty-four new quadratic transformations of hypergeometric functions, and obtain several new summation theorems.

Keywords: Fractional derivatives, Well-Poised Fractional Calculus Operator, Special functions, Gauss hypergeometric function, Transformation formulas

2010 MSC: 26A33, 33C05, 33C20

1. Introduction and motivation

The fractional derivative of arbitrary complex order α , noted $D^{\alpha}_{g(z)}F(z)$, is an extension of the familiar nth derivative $D^{n}_{g(z)}F(z) = d^nF(z)/d(g(z))^n$ of the function $F(z)$ with respect to $g(z)$ for non-integral values of n. We can find many surveys and discussions on this subject in the literature [1, 2, 21, 22, 25]. A large bibliography can be found in [25]. The fractional derivative is a powerful tool for generalizing classical results on the nth derivative. Most of the properties of the classical calculus and a large number of formulas from the elementary calculus have been extended to the fractional calculus. For instance, we have the composition rule [13], the Leibniz rule [11, 12, 27, 28], the chain rule [11], and Taylor's and Laurent's series [15, 25, 29].

The most familiar representation for the fractional derivative of order α of $f(z)$ is the Riemann-Liouville integral [10]

$$D^{\alpha}_{z}z^pf(z) = \frac{1}{\Gamma(-\alpha)}\int_0^z \xi^p f(\xi)(\xi-z)^{-\alpha-1}\,\mathrm{d}\xi \tag{1.1}$$
$$(Re(\alpha) < 0\,; Re(p) > -1)$$

†*Article ID:* MTJPAM-D-20-00005
Email address: rtrembla@uqac.ca (R. Tremblay)
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⋆*Correspending Author:* R. Tremblay

where the integration is carried out along a straight line from 0 to z in the complex ξ -plane. Integrating by parts m times, we obtain

$$D_z^{\alpha} z^p f(z) = \frac{d^m}{dz^m} D_z^{\alpha-m} z^p f(z). \tag{1.2}$$

This allows us to modify the restriction $\Re(\alpha) < 0$ to $\Re(\alpha) < m$ [20].

Another representation for the fractional derivative $D_{g(z)}^{\alpha}[g(z)]^p f(z)$ (where $g(z)$ is a regular univalent function) is based on the Cauchy integral formula, allows us to reduce $\Re(\alpha) < 0$ to $\alpha \neq -1, -2, -3, \ldots$.

$$\begin{aligned} & D_z^{\alpha}[g(z)]^p f(z) \\ &= \frac{\Gamma(1+\alpha)}{2\pi i} \int_{g^{-1}(0)}^{z^+} [g(\xi)]^p f(\xi) g'(\xi)[g(\xi) - g(z)]^{-\alpha-1} d\xi \end{aligned} \tag{1.3}$$

where the contour of integration starts at $g^{(-1)}(0)$, circles z once in the positive direction, and returns to $g^{(-1)}(0)$. This representation has been widely used in many interesting papers (for example, [11, 12, 13, 14, 15, 16, 17, 18]).

A less restrictive representation of the fractional derivative is one based on the Pochhammer contour integral introduced by Tremblay [24] and Tremblay *et al.* [8, 9].

Definition 1.1. Let $f(z)$ be analytic in a simply connected region of $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$, and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$. Let $F(a) = f(a)\, g(a)^p (g(a) - g(z))^{-\alpha-1}$ denote the principal value. Then if α is not a negative integer, p is not an integer, and $z \in \mathcal{R} \setminus \{g^{-1}(0)\}$, we define the fractional derivative of order α of $g(z)^p f(z)$ with respect to $g(z)$ to be

$$\begin{aligned} D_{g(z)}^{\alpha} \{[g(z)]^p f(z)\} = {} & \frac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi \sin(\pi p)} \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \\ & \cdot \frac{f(\xi)[g(\xi)]^p g'(\xi)}{[g(\xi) - g(z)]^{\alpha+1}}\, d\xi. \end{aligned} \tag{1.4}$$

For non-integers α and p, the functions $g(\xi)^p$ and $[g(\xi) - g(z)]^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi = z$ and $\xi = g^{-1}(0)$, and both branches pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = C_1 \cup C_2 \cup C_3 \cup C_4$ at any other point, as shown in Figure **1**. Here $F(a)$ denotes the principal value of the integrand in (1.4) at the beginning and the ending point of the Pochhammer contour P(a), which is closed on the Riemann surface of the multiple-valued function $F(\xi)$.

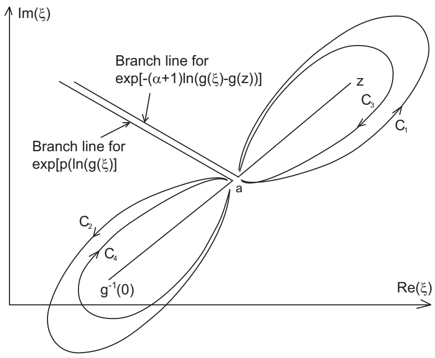

Figure 1. *The contour used in integral* (1.4).

Remark 1.2. In Definition 1.1, the function $f(z)$ must be analytic at $\xi = g^{-1}(0)$. However, it is interesting to note here that, if we could also allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$, then (1.4) would still be valid.

Remark 1.3. Since the Pochhammer contour never crosses the singularities at $\xi = g^{-1}(0)$ and $\xi = z$ in (1.4), then we know that the integral is analytic for all p and for all α and for $z \in \mathcal{R}\backslash\{g^{-1}(0)\}$. Indeed, in this case, the only possible singularities of $D^{\alpha}_{g(z)}[g(z)]^p f(z)$ are $\alpha = -1, -2, \dots$ and $p = 0, \pm 1, \pm 2, \dots$, which can directly be identified from the coefficient of the integral (1.4). However, integrating by parts N times the integral in (1.4) by two different ways, we can show that $\alpha = -1, -2, \dots$ and $p = 0, 1, 2, \dots$ are removable singularities (for details, see [8]).

Adopting the Pochhammer based representation for the fractional derivative reduces the restriction to the case when p is not a negative integer. However, $D^{\alpha}_{g(z)} \dfrac{[g(z)]^p}{\Gamma(1+p)} f(z)$ is valid for all values of p.

2. The well-poised fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$

Now let us recall the well-poised fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$. This operator was introduced by Tremblay [24] and is defined in terms of the fractional calculus operator $D^{\alpha}_{g(z)}$ as

$$ {}_{g(z)}O^{\alpha}_{\beta} f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} g(z)^{1-\beta} D^{\alpha-\beta}_{g(z)} g(z)^{\alpha-1} f(z). \tag{2.1} $$

This equation seems to be simply a rewrite of the fractional derivative definition (1.4). However, it is motivated because it has an important list of easy-to-demonstrate properties that simplify calculations while having simpler analytical properties than the fractional derivative itself. It is more appropriate to explore special functions as was already demonstrated in [4, 5, 24, 26]. These are the reasons why we called the 'well-poised fractional calculus operator'.

The operator ${}_{g(z)}O^{\alpha}_{\beta}$ possesses the following integral representation with regard to the Pochhammer integral contour.

Definition 2.1. Let $f(z)$ be analytic in a simply connected region of $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$, and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$. Let $F(a) = f(a)\, g(a)^{\alpha-1}(g(a) - g(z))^{\beta-\alpha-1}$ denote the principal value. Then if $\alpha \neq 1, 2, \dots, \beta - \alpha \neq 1, 2, \dots, \beta \neq 0, -1, -2 \dots$ and $z \in \mathcal{R} \setminus \{g^{-1}(0)\}$, we define the fractional operator ${}_{g(z)}O^{\alpha}_{\beta}$ with parameters α and β on $f(z)$ with respect to $g(z)$ to be

$$ \begin{aligned} {}_{g(z)}O^{\alpha}_{\beta} f(z) = &-\frac{e^{-i\pi\beta}\Gamma(\beta)}{4\pi \sin(\pi\alpha)\sin(\pi(\beta-\alpha))} \\ &\cdot \int_{C(z^+, g^{-1}(0)^+, z^-, g^{-1}(0)^-; F(a), F(a))} \frac{f(\xi)\,[g(\xi)]^{\alpha-1}\, g'(\xi)}{[g(\xi) - g(z)]^{\alpha-\beta+1}}\, d\xi. \end{aligned} \tag{2.2} $$

Remark 2.2. The restriction in Definition 2.1 derives essentially from the product of the three gamma functions $\Gamma(\beta)\Gamma(1-\alpha)\Gamma(1+\alpha-\beta)$ in (2.2). However, as mentioned in Definition 2.1 of the fractional derivative, integrating by parts N times the integral in (2.2) two different ways, we can show that $\alpha = 1, 2, \dots,$ and $\beta - \alpha = 1, 2, \dots$ are removable singularities [9]. Moreover, the operator $\frac{{}_{g(z)}O^{\alpha}_{\beta}}{\Gamma(\beta)}$ no longer has the restriction on the parameter β and becomes well defined for all values of α and β. Furthermore, the function $f(z)$ can have an essential singularity at $z = g^{-1}(0)$.

An important feature of Definition 2.1 is the symmetry of the Pochhammer contour used around points of singularities $g^{(-1)}(0)$ and z. By a simple change of variables $\zeta = z - \xi$ in (2.2), the author [24, 30] deduces the following transformation formula for the fractional operator ${}_{g(z)}O^{\alpha}_{\beta}$.

Theorem 2.3. *Let $f(z)$ be a function that satisfies the conditions for the existence of the fractional derivative ${}_{g(z)}O^{\alpha}_{\beta} f(z)$ listed in Definition 2.1(see Figure 1). If $f(g^{-1}(0)) \neq 0$ and $\beta \neq 0, -1, -2, \dots$ then we have*

$$ {}_{g(z)}O^{\alpha}_{\beta} f(z) \quad = \quad {}_{g(z)}O^{\beta-\alpha}_{\beta} f(g^{-1}(g(w) - g(z)))\Big|_{w=z}, \tag{2.3} $$

for $z \in \mathcal{R} \setminus \{g^{-1}(0)\}$. Note that we must have $w \to z$ in the right side of (2.3) after evaluation of the fractional derivative, and therefore the point w must be near the point z inside the loop C_3.

In terms of the fractional derivative, the transformation (2.3) becomes

$$\begin{aligned} & D^{\alpha}_{g(z)} \{[g(z)]^p f(z)\} \\ & = \frac{\Gamma(1+p)}{\Gamma(-\alpha)} D^{-p-1}_{g(z)} g(z)^{-\alpha-1} f(g^{-1}(g(w)-g(z)))\Big|_{w=z}. \end{aligned} \tag{2.4}$$

Several articles [4, 5, 24, 30] show the effectiveness of these operators using the generalized chain rule (2.5) for the fractional derivative given in terms of the well-poised fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$ given in the following theorem. It plays a key role in Sections **4** and **5** in finding new transformation formulas involving special functions with quadratic arguments. In a future article, the same method will be extended to special functions with higher degree arguments.

Theorem 2.4. *Let $f(g^{-1}(z))$ and $f(h^{-1}(z))$ be defined and analytic on the simply connected region $\mathcal{R}$. Let $f(z)$ be a function that satisfies the conditions for the existence of ${}_{g(z)}O^{\alpha}_{\beta} f(z)$ and ${}_{h(z)}O^{\alpha}_{\beta} f(z)$ listed in Definition 2.1 and using a Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ laid out around the points $\xi = g^{-1}(0)$, $\xi = h^{-1}(0)$ and z (see Figure **2**). For $z \in \mathcal{R} \setminus \{g^{-1}(0), h^{-1}(0)\}$ we have*

$$\begin{aligned} & {}_{g(z)}O^{\alpha}_{\beta} f(z) \\ & = \left(\frac{g(z)}{h(z)}\right)^{1-\beta} \left\{ {}_{h(z)}O^{\alpha}_{\beta} \left(\frac{g(z)}{h(z)}\right)^{\alpha-1} \frac{g'(z)}{h'(z)} \left(\frac{g(z)-g(w)}{h(z)-h(w)}\right)^{\beta-\alpha-1} f(z) \right\}\Bigg|_{w=z}. \end{aligned} \tag{2.5}$$

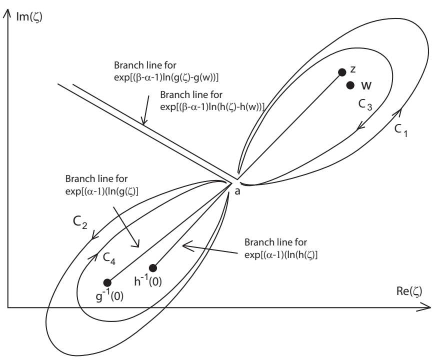

Figure 2. *The contour used in integral* (2.5).

3. Some properties of the Fractional Calculus Operator ${}_{g(z)}O^{\alpha}_{\beta}$.

In this section, we recall some of the important properties of the fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$ (see [24]). We chose to simply list them since the proofs are readily obtainable.

Property 3.1. *Linearity:*

$${}_{g(z)}O^{\alpha}_{\beta}\{\lambda_1 f(z) + \lambda_2 h(z)\} = \lambda_1\, {}_{g(z)}O^{\alpha}_{\beta} f(z) + \lambda_2\, {}_{g(z)}O^{\alpha}_{\beta} h(z) \tag{3.1}$$

where λ_1 and λ_2 are arbitrary complex numbers.

Property 3.2. *Invariability:*

$$ {}_{\lambda g(z)}O_{\beta}^{\alpha} f(z) = {}_{g(z)}O_{\beta}^{\alpha} f(z) \tag{3.2} $$

where $\lambda \neq 0$ is an arbitrary complex number.

Property 3.3. *Identity:*

$$ {}_{g(z)}O_{\alpha}^{\alpha} = I. \tag{3.3} $$

Property 3.4. *Reductions:*

$$ {}_{g(z)}O_{\beta}^{\alpha} \;\; {}_{g(z)}O_{\gamma}^{\beta} = \;\; {}_{g(z)}O_{\gamma}^{\alpha}, \tag{3.4} $$

$$ {}_{g(z)}O_{\beta}^{\alpha} \;\; {}_{g(z)}O_{\alpha}^{\gamma} = \;\; {}_{g(z)}O_{\beta}^{\gamma}. \tag{3.5} $$

Property 3.5. *Elementary cases:*

$$ {}_{g(z)}O_{\beta}^{\alpha}\, 1 = 1, \tag{3.6} $$

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(z)]^n = \frac{(\alpha)_n}{(\beta)_n}\, [g(z)]^n, \tag{3.7} $$

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(w) - g(z)]^n \Big|_{w=z} = \frac{(\beta-\alpha)_n}{(\beta)_n}\, [g(z)]^n, \tag{3.8} $$

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(z)]^n [g(w) - g(z)]^m \Big|_{w=z} = \frac{(\alpha)_n (\beta-\alpha)_m}{(\beta)_{n+m}}\, [g(z)]^{n+m}, \tag{3.9} $$

where m and n are integers and $(\lambda)_k$ holds for the Pochhammer symbol defined, in terms of the gamma function, by

$$ (\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N};\, \lambda \in \mathbb{C}) \\ 1 & (k = 0;\, \lambda \in \mathbb{C} \setminus \{0\}). \end{cases} $$

Property 3.6. *Useful cases:*

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(z)]^{\lambda} f(z) = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\beta+\lambda)}\, [g(z)]^{\lambda}\, {}_{g(z)}O_{\beta+\lambda}^{\alpha+\lambda}\, f(z), \tag{3.10} $$

$$ {}_{g(z)}O_{\beta}^{\alpha}\, [g(w) - g(z)]^{\theta} f(z) \Big|_{w=z} = \frac{\Gamma(\beta)\Gamma(\beta-\alpha+\theta)}{\Gamma(\beta-\alpha)\Gamma(\beta+\theta)}\, [g(z)]^{\theta}\, {}_{g(z)}O_{\beta+\theta}^{\alpha}\, f(z). \tag{3.11} $$

Property 3.7. *Combined case:*

$$ \begin{aligned} &{}_{g(z)}O_{\beta}^{\alpha}\, [g(z)]^{\lambda} [g(w) - g(z)]^{\theta} f(z) \Big|_{w=z} \\ &\qquad = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)\Gamma(\beta-\alpha+\theta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)\Gamma(\beta+\lambda+\theta)}\, [g(z)]^{\lambda+\theta}\, {}_{g(z)}O_{\beta+\lambda+\theta}^{\alpha+\lambda}\, f(z). \end{aligned} \tag{3.12} $$

Property 3.8. *Commutativity:*

$$ {}_{g(z)}O_{\beta}^{\alpha}\, {}_{g(z)}O_{\theta}^{\delta} = {}_{g(z)}O_{\theta}^{\delta}\, {}_{g(z)}O_{\beta}^{\alpha}, \tag{3.13} $$

$$ \begin{aligned} &{}_{g(z)}O_{\beta}^{\alpha}\;\, g(z)^{\gamma}\;\, {}_{g(z)}O_{\theta}^{\delta} \\ &= \frac{\Gamma(\beta)\Gamma(\theta)\Gamma(\alpha+\gamma)\Gamma(\delta-\gamma)}{\Gamma(\alpha)\Gamma(\delta)\Gamma(\beta+\gamma)\Gamma(\theta-\gamma)}\, {}_{g(z)}O_{\theta-\gamma}^{\delta-\gamma}\;\, g(z)^{\gamma}\;\, {}_{g(z)}O_{\beta+\gamma}^{\alpha+\gamma}. \end{aligned} \tag{3.14} $$

Property 3.9. *Effect on a hypergeometric function:*

$$ {}_{g(z)}O_{\beta}^{\alpha}\;\, {}_{p}F_{q}\left[\left.\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix}\right| g(z)\right] = {}_{p+1}F_{q+1}\left[\left.\begin{matrix} \alpha, a_1, a_2, \cdots, a_p \\ \beta, b_1, b_2, \cdots, b_q \end{matrix}\right| g(z)\right] \tag{3.15} $$

where

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix}\middle| g(z)\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{[g(z)]^n}{n!}. $$

in which no denominator parameter b_i ($i = 1, 2, \ldots, q$) is allowed to be zero or a negative integer.

Property 3.10. *Contiguous relations:*

$$ (\alpha-\beta)\ {}_{g(z)}O_{\gamma}^{\alpha}\ {}_{g(z)}O_{\delta}^{\beta} = \ \alpha\ {}_{g(z)}O_{\gamma}^{\alpha+1}\ {}_{g(z)}O_{\delta}^{\beta}\ -\ \beta\ {}_{g(z)}O_{\gamma}^{\alpha}\ {}_{g(z)}O_{\delta}^{\beta+1}, \tag{3.16} $$

$$ (\alpha-\beta+1)\ \ {}_{g(z)}O_{\beta}^{\alpha}\ =\ \alpha\ {}_{g(z)}O_{\beta}^{\alpha+1}\ \ -\ \ (\beta-1)\ {}_{g(z)}O_{\beta-1}^{\alpha}. \tag{3.17} $$

Property 3.11. *Special cases:*

$$ {}_zO_{\beta}^{-n} f(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f^{(k)}(0) \frac{z^k}{(\beta)_k}, \tag{3.18} $$

$$ {}_zO_{\alpha}^{\alpha+n} f(z) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) \frac{z^k}{(\alpha)_k}, \tag{3.19} $$

$$ \begin{aligned} {}_zO_{-m}^{-n} f(z) &= \sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{m}{k}} f^{(k)}(0) \frac{z^k}{k!} \\ &\quad + \lim_{\epsilon\to 0} \sum_{k=m+1}^{\infty} \frac{(-n+\epsilon)_k}{(-m+\epsilon)_k} f^{(k)}(0) \frac{z^k}{k!} \qquad m \geq n. \end{aligned} \tag{3.20} $$

Property 3.12. *Particular transformations:*

$$ {}_{\frac{z}{1-z}}O_{\beta}^{\alpha} f(z) \ = \ (1-z)^{\alpha}\, {}_zO_{\beta}^{\alpha} (1-z)^{-\beta} f(z), \tag{3.21} $$

$$ {}_{\frac{az+b}{cz+d}}O_{\beta}^{\alpha} f(z) \ = \ (\frac{a}{ad-bc})^{\alpha-\beta} (cz+d)^{\alpha}\ {}_{az+b}O_{\beta}^{\alpha} (cz+d)^{-\beta} f(z), \tag{3.22} $$

$$ \begin{aligned} &{}_zO_{\beta}^{\alpha} (1-z)^{\theta} f(z) \\ &= (1-z)^{\beta-\alpha+\theta}\, {}_zO_{\beta}^{\beta+\theta} (1-z)^{\alpha-\beta}\, {}_zO_{\beta+\theta}^{\alpha} f(z). \end{aligned} \tag{3.23} $$

Property 3.13. *Logarithmic functions:*

$$ \begin{aligned} {}_zO_{\beta}^{\alpha}\, z^{\mu}(w-z)^{\gamma}(ln(z))^{\delta}(ln(w-z))^{\theta}\Big|_{w=z}^{*} &= \frac{\Gamma(\beta)\Gamma(\alpha+\mu)\Gamma(\beta-\alpha+\gamma)}{\Gamma(\alpha)\Gamma(\beta-\alpha)\Gamma(\beta+\mu+\gamma)} z^{\mu+\gamma} \\ \Big\{[\psi(\alpha+\mu)-\psi(\beta+\mu+\gamma)+\ln(z)]^{\delta}&[\psi(\beta-\alpha+\gamma)-\psi(\beta+\mu+\gamma)+\ln(z)]^{\theta} \\ &\qquad -\delta\theta\psi'(\beta+\mu+\gamma)\Big\} \end{aligned} \tag{3.24} $$

with δ, $\theta = 0$ or 1 and $\psi(z) = \Gamma(z)^{'} / \Gamma(z)$ is the Psi (or Digamma) function.

We mention that ${}_{g(z)}O_{\beta}^{\alpha}\{(g(z))^p\{\ln(g(z))\}^{\delta} f(z)\big|_{w=z}$ (with $\delta = 0$ or 1) has many more interesting properties and applications. The analytical properties with respect to parameters α, β, p and z were studied in depth in [8, 24] where several kinds of applications to special functions have been explored. An article about the more general operator ${}_{g(z)}O_{\beta}^{\alpha}F(w,z)\big|_{w=z}$ where $F(w,z) = ((g(z))^p(g(w)-g(z))^q\{\ln(g(z))\}^{\delta}\{\ln(g(w))-g(z)\}^{\delta} f(z)$ is in preparation.

The effectiveness of this operator in studying special functions and finding new hypergeometric transformations is illustrated in the next Section.

4. Some applications to particular transformation formulas involving a Gauss hypergeometric function.

Recently, Tremblay and Gaboury [26] applied this new method for the first time, which allowed to obtain several formulas of transformation with rational (quadratic, cubic and of higher degree) arguments. For example, from the following quadratic transformation obtained by Goursat [6, Eq. (44), p. 120]), we find the following two presumably new transformation formulas

$$(1-z)^{1-a/2}{}_2F_1\left[\begin{matrix}a/2-1, b-a/2\\ b+1/2\end{matrix}\middle| -\frac{z^2}{4(1-z)}\right] = {}_3F_2\left[\begin{matrix}a-2, a, b\\ a-1, 2b\end{matrix}\middle| z\right] - \frac{(a-2)}{2(a-1)}z\ {}_2F_1\left[\begin{matrix}a-1, b\\ 2b\end{matrix}\middle| z\right] \tag{4.1}$$

and

$$\begin{aligned}(1-z)^{2-a/2}{}_2F_1\left[\begin{matrix}a/2-2, b-a/2\\ b+1/2\end{matrix}\middle| -\frac{z^2}{4(1-z)}\right] &= \frac{(a-2)}{2(a-3)}\,{}_3F_2\left[\begin{matrix}a-4, a, b\\ a-2, 2b\end{matrix}\middle| z\right] + (2-3z)\frac{(a-4)}{4(a-3)}\,{}_3F_2\left[\begin{matrix}a-3, a, b\\ a-1, 2b\end{matrix}\middle| z\right]\\ &\quad - \frac{(a-4)}{4(a-1)}z(1-z)\ {}_2F_1\left[\begin{matrix}a-2, b\\ 2b\end{matrix}\middle| z\right].\end{aligned} \tag{4.2}$$

Remark 4.1. The restrictions on the parameters of these formulas include those which allow the hypergeometric functions to be well defined. In particular, no denominator parameter in any hypergeometric function involved is allowed to be zero or one negative integer. Additional restrictions are often easy to deduce. They ensure the existence of the coefficients appearing in these formulas. This remark applies to all the formulas in this paper.

The new transformations often make it possible to discover new summation theorems for hypergeometric functions. For example, this was done recently by Choi and Rathie [3] who deduced several summation formulas from transformations formulas for the Kampé de Fériet function established by Liu and Wang [7].

As example, if $b \to 0$ in (4.1), using (3.17) we obtain after simplifications

$$\begin{aligned}(1-z)^{1-a/2}{}_2F_1\left[\begin{matrix}a/2-1, -a/2\\ 1/2\end{matrix}\middle| -\frac{z^2}{4(1-z)}\right] &= \frac{1}{4(a-1)(1-z)}\\ &\quad \{(1-z)^{2-a}(2(a-1)-az) + (a-2)z^2 - (3a-4)z + 2(a-1)\}.\end{aligned} \tag{4.3}$$

Applying the operator ${}_zO_\beta^\alpha\{\}\big|_{z=1}$ on both sides of (4.3) and using the fact that

$${}_zO_\beta^\alpha\left\{\frac{z^{2n}}{(1-z)^{a/2-1+n}}\right\}\bigg|_{z=1} = \frac{\Gamma(\beta)\Gamma(\beta-\alpha-\frac{a}{2}+1)}{\Gamma(\beta-\alpha)\Gamma(\beta-\frac{a}{2}+1)}\frac{(\frac{\alpha}{2})_n(\frac{\alpha}{2}+\frac{1}{2})_n(-1)^n 4^n}{(\beta-\frac{a}{2}+1)_n(\alpha-\beta+\frac{a}{2})_n}, \tag{4.4}$$

we obtain the following presumed new summation theorem:

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}-a/2, a/2-1, \alpha/2, \alpha/2+1/2\\ 1/2, \alpha-\beta+a/2, \beta-a/2+1\end{matrix}\middle| 1\right]\\ &= \frac{\Gamma(\beta-a/2+1)}{4(a-1)\Gamma(\beta-\alpha-a/2+1)}\left\{\frac{\Gamma(\beta-\alpha-a+1)}{\Gamma(\beta-a+2)}\big[2(a-1)(\beta-a+1)-a\alpha\big]\right.\\ &\left.+ \frac{\Gamma(\beta-\alpha-1)}{\Gamma(\beta+1)}\big[(a-2)\alpha(\alpha+1)-(3a-4)\alpha\beta+2(a-1)\beta(\beta-1)\big]\right\}.\end{aligned} \tag{4.5}$$

If $\beta \to \alpha+1$ in (4.5), we obtain

$${}_3F_2\left[\begin{matrix}-a/2, \alpha/2, \alpha/2+1/2\\ 1/2, \alpha-a/2+2\end{matrix}\middle| 1\right] \tag{4.6}$$

$$= \frac{\Gamma(2-a)\Gamma(\alpha+2)(2(a-1)(\alpha-a+2)-a\alpha)+(a\alpha+2a-2)\Gamma(\alpha-a+3)}{\Gamma(\alpha-a+3)\Gamma(\alpha+2)}.$$

Equations (4.6) and (4.5) are valid if α is a negative integer or zero since the point $z = 1$ is outside the convergence region of the hypergeometric function in (4.3).

Now, we present some examples using the fractional operator ${}_{g(z)}O^{\alpha}_{\beta}$ to discover new formulas involving hypergeometric functions. To make the document self-contained, we also illustrate the method used to obtain all formulas listed in Section **5**. By using several properties of the well-formed fractional calculator ${}_{g(z)}O^{\alpha}_{\beta}$, we also obtain some results arising from this presumed new formula. Each of the formulas of the list could be explored in the same way to obtain new results such as formulas of transformation, theorems of addition, integral representations, and so on.

Example 4.2. From the following well-known quadratic transformation holding for the Gauss hypergeometric function [6, Eq. (45), p. 120]:

$${}_2F_1\left[\begin{matrix}a,b\\2b\end{matrix}\middle|z\right] = (1-z/2)^{-a}\ {}_2F_1\left[\begin{matrix}a/2, a/2+1/2\\b+1/2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right], \tag{4.7}$$

we can obtain the next transformation formulas

$${}_2F_1\left[\begin{matrix}a/2, a/2-1/2-n\\b+1/2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] = 2^{1-a}(2-z)^{a-2n-1}\frac{(1-a)_n}{(1-a/2)_n} \tag{4.8}$$
$$\sum_{k=0}^{n}\binom{n}{k}(1-z)^k\frac{(a-1-2n)_k}{(a-n)_k}\ {}_3F_2\left[\begin{matrix}a-1-2n+k, a, b\\a-n+k, 2b\end{matrix}\middle|z\right].$$

Proof of (4.8). Setting $h(z) = z$ and $g(z) = \left(\frac{z}{2-z}\right)^2$ in (2.5), then $\frac{g'(z)}{h'(z)} = \frac{4z}{(2-z)^3}$, $\frac{g(z)-g(w)}{z-w} = 4\frac{(z+w-zw)}{(z-2)^2(w-2)^2}$. We obtain

$${}_{(\frac{z}{2-z})^2}O^{\alpha}_{\beta}f(z) \tag{4.9}$$
$$= 2^{2\beta-2\alpha}z^{1-\beta}(2-z)^{2\alpha}\ {}_zO^{\alpha}_{\beta}\left\{z^{\alpha}(2-z)^{1-2\beta}(w+z-wz)^{\beta-\alpha-1}f(z)\right\}\Big|_{w=z}.$$

In (4.9), setting $\beta = \alpha+n+1$, $f(z) = {}_2F_1\left[\begin{matrix}a/2, a/2+1/2\\b+1/2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right]$ in the l. h. s. , $f(z) = (1-z/2)^a\ {}_2F_1\left[\begin{matrix}a,b\\2b\end{matrix}\middle|z\right]$ in the r. h. s. and using the fact that $(w+z-wz)^n = \sum_{k=0}^{n}\binom{n}{k}w^{n-k}(1-w)^kz^k$, we get

$$2^{a-2n-2}(2-z)^{-2\alpha}z^{\alpha+n}\ {}_{(\frac{z}{2-z})^2}O^{\alpha}_{\alpha+n+1}\ {}_2F_1\left[\begin{matrix}a/2, a/2+1/2\\b+1/2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] \tag{4.10}$$
$$= {}_zO^{\alpha}_{\alpha+n+1}\left\{(2-z)^{-2\alpha+a-2n-1}z^{\alpha}(w+z(1-w))^n\ {}_2F_1\left[\begin{matrix}a,b\\2b\end{matrix}\middle|z\right]\right\}\Big|_{w=z}$$
$$= \sum_{k=0}^{n}\binom{n}{k}z^{n-k}(1-z)^k\ {}_zO^{\alpha}_{\alpha+n+1}\ z^{\alpha+k}(2-z)^{-2\alpha+a-2n-1}\ {}_2F_1\left[\begin{matrix}a,b\\2b\end{matrix}\middle|z\right].$$

Putting $\alpha = a/2-1/2-n$ and again using properties (3.10) and (3.15), we obtain the result

$$2^{a-2n-2}(2-z)^{2n+1-a}\ {}_2F_1\left[\begin{matrix}a/2, a/2-1/2-n\\b+1/2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] \tag{4.11}$$
$$= \sum_{k=0}^{n}\binom{n}{k}(1-z)^k\ \frac{\Gamma(a/2+1/2)\Gamma(a-2-2n+k)}{\Gamma(a/2-1/2-n)\Gamma(a-n+k)}\ {}_3F_2\left[\begin{matrix}a+2n-1+k, a, b\\a-n+k, 2b\end{matrix}\middle|z\right]$$

which is equivalent to (4.8). □

Note that the cases $n = 1$, $n = 2$ and $n = 3$ in (4.11) give the following transformation formulas

$$2^{a-2}(2-z)^{3-a}\,{}_2F_1\left[\begin{matrix}a/2-3/2, a/2\\ b+1/2\end{matrix}\middle|\Big(\frac{z}{2-z}\Big)^2\right] \tag{4.12}$$
$$= \frac{(a-1)}{(a-2)}\,{}_3F_2\left[\begin{matrix}a-3, a, b\\ a-1, 2b\end{matrix}\middle| z\right] + \frac{(a-3)}{(a-2)}(1-z)\,{}_2F_1\left[\begin{matrix}a-2, b\\ 2b\end{matrix}\middle| z\right],$$

$$2^{a-3}(2-z)^{5-a}{}_2F_1\left[\begin{matrix}a/2, a/2-5/2\\ b+1/2\end{matrix}\middle|\Big(\frac{z}{2-z}\Big)^2\right]$$
$$= \frac{(a-1)}{(a-4)}{}_3F_2\left[\begin{matrix}a-5, a, b\\ a-2, 2b\end{matrix}\middle| z\right] + 2(1-z)\frac{(a-1)(a-5)}{(a-2)(a-4)}{}_3F_2\left[\begin{matrix}a-4, a, b\\ a-1, 2b\end{matrix}\middle| z\right]$$
$$+ (1-z)^2\frac{(a-5)}{(a-2)}{}_2F_1\left[\begin{matrix}a-3, b\\ 2b\end{matrix}\middle| z\right] \tag{4.13}$$

and

$$2^{a-4}(2-z)^{7-a}{}_2F_1\left[\begin{matrix}a/2, a/2-7/2\\ b+1/2\end{matrix}\middle|\Big(\frac{z}{2-z}\Big)^2\right] \tag{4.14}$$
$$= \frac{(a-1)(a-3)}{(a-4)(a-6)}{}_3F_2\left[\begin{matrix}a-7, a, b\\ a-3, 2b\end{matrix}\middle| z\right] + 3(1-z)\frac{(a-1)(a-7)}{(a-4)(a-6)}{}_3F_2\left[\begin{matrix}a-6, a, b\\ a-2, 2b\end{matrix}\middle| z\right]$$
$$+ 3(1-z)^2\frac{(a-1)(a-7)}{(a-2)(a-4)}{}_3F_2\left[\begin{matrix}a-5, a, b\\ a-1, 2b\end{matrix}\middle| z\right]$$
$$+ (1-z)^3\frac{(a-5)(a-7)}{(a-2)(a-4)}{}_2F_1\left[\begin{matrix}a-4, b\\ 2b\end{matrix}\middle| z\right]. \tag{4.15}$$

If we put $z = 1$, (4.11) becomes

$$2^{a-2n-2}\,{}_2F_1\left[\begin{matrix}a/2, a/2-1/2-n\\ b+1/2\end{matrix}\middle| 1\right] \tag{4.16}$$
$$= \frac{\Gamma(a/2+1/2)\Gamma(a-2-2n)}{\Gamma(a/2-1/2-n)\Gamma(a-n)}{}_3F_2\left[\begin{matrix}a+2n-1, a, b\\ a-n, 2b\end{matrix}\middle| 1\right].$$

With the Gauss summation

$$ {}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}\middle| 1\right] = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}, \tag{4.17}$$

we deduce

$$ {}_3F_2\left[\begin{matrix}a-1+2n, a, b\\ a-n, 2b\end{matrix}\middle| 1\right] \tag{4.18}$$
$$= \frac{\Gamma(b+1/2)\Gamma(1/2)\Gamma(a-n)\Gamma(1+b-a+n)}{\Gamma(a/2-n)\Gamma(a/2+1/2)\Gamma(1/2+b-a/2)\Gamma(1+b-a/2+n)}$$

which is presumably new.

We can also write $(w+z-wz)^n = \sum_{k=0}^{n}\binom{n}{k}z^{n-k}(1-z)^k w^k$ in (4.9). Again, if we put $\beta = \alpha + n + 1$ and with $f(z) = (1-(z/(2-z)^2)^{-\gamma} = (1-z)^{-\gamma}(1-z/2)^{2\gamma}$, using the fact that [9, Appell's functions, p. 260]

$$F_1(a; b, b'; c; xt, yt) = \frac{\Gamma(c)}{\Gamma(a)}t^{1-c}D_t^{c-a}t^{a-1}(1-xt)^b(1-yt)^{b'},$$

$$= {}_tO_c^a(1-xt)^b(1-yt)^{b'}, \tag{4.19}$$

we obtain

$$\begin{aligned} &{}_2F_1\left[\begin{matrix}\alpha,\gamma\\ \alpha+n+1\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] \\ &= \frac{(\alpha+1)_n(2\alpha)_n}{(1+2\alpha)_{2n}}(1-z/2)^{2\alpha}\sum_{k=0}^{n}\binom{n}{k}\frac{(-2\alpha-2n)_k}{(1-2\alpha-n)_k} \\ &\qquad F_1(2\alpha+n-k;1-2\gamma+2\alpha+2n,\gamma-k;2\alpha+2n-k+1;z/2,z). \end{aligned} \tag{4.20}$$

If $n=0$ in (4.20), we find

$$F_1(2\alpha;1-2\gamma+2\alpha,\gamma;2\alpha+1;z/2,z) = (1-z/2)^{-2\alpha}{}_2F_1\left[\begin{matrix}\alpha,\gamma\\ \alpha+1\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right]. \tag{4.21}$$

Each of the formulas obtained has the potential for the discovery of new results. For example, if $b\to 0$ in (4.12), we get

$$\begin{aligned} &2^{a-2}(2-z)^{3-a}\,{}_2F_1\left[\begin{matrix}a/2-3/2,a/2\\ 1/2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] \\ &= \frac{(a-1)}{(a-2)}\left(\frac{1}{2}+\frac{1}{2}{}_2F_1\left[\begin{matrix}a-3,a\\ a-1\end{matrix}\middle|z\right]\right)+\frac{(a-3)}{(a-2)}(1-z)\left(\frac{1}{2}+\frac{1}{2}(1-z)^{2-a}\right). \end{aligned} \tag{4.22}$$

Now, using (2.3) with $g(z)=z$, we can write

$$\begin{aligned} &{}_2F_1\left[\begin{matrix}a-3,a\\ a-1\end{matrix}\middle|z\right] = {}_zO_{a-1}^a\,(1-z)^{3-a} = {}_zO_{a-1}^{-1}\,(1-w+z)^{3-a}\Big|_{w=z} \\ &= (1-z)^{3-a}{}_zO_{a-1}^{-1}\,(1+\frac{z}{1-w})^{3-a}\Big|_{w=z} = (1-z)^{3-a}\sum_{n=0}^{\infty}\frac{(a-3)_n(-1)^n}{n!(1-z)^n}{}_zO_{a-1}^{-1}\,z^n \\ &= (1-z)^{3-a}\sum_{n=0}^{1}\frac{(a-3)_n(-1)_n(-1)^nz^n}{n!(a-1)_n(1-z)^n} = (1-z)^{2-a}\frac{(a-1-2z)}{a-1}. \end{aligned} \tag{4.23}$$

With (4.23), we easily get

$$\begin{aligned} &(2-z)^{3-a}\,{}_2F_1\left[\begin{matrix}a/2-3/2,a/2\\ 1/2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] \\ &= \frac{2^{1-a}}{(a-2)}\Big[(a-1)+(a-1)(1-z)^{3-a}+(a-3)(1-z)^{2-a}+(a-3)(1-z)\Big]. \end{aligned} \tag{4.24}$$

Replacing z by $2z$, putting $a=3-2n$ and applying the operator ${}_zO_{2\alpha}^{\alpha}\{\}\big|_{z=1}$ on each side of the equation (4.24), we obtain after simplifications

$$\begin{aligned} &{}_4F_3\left[\begin{matrix}-n,3/2-n,\alpha/2,\alpha/2+1/2\\ 1/2,-\alpha/2-n+1/2,-\alpha/2-n+1\end{matrix}\middle|1\right] \\ &= \frac{(2\alpha)_{2n}}{2(2n-1)(\alpha)_{2n}}\left\{(2n-1)\,{}_2F_1\left[\begin{matrix}\alpha,-2n\\ 2\alpha\end{matrix}\middle|2\right]+n\,{}_2F_1\left[\begin{matrix}\alpha+1,-2n+1\\ 2\alpha+1\end{matrix}\middle|2\right]+n-1\right\}. \end{aligned} \tag{4.25}$$

With the help of (3.17), we can easily show that

$${}_2F_1\left[\begin{matrix}\alpha+1,-2n+1\\ 2\alpha+1\end{matrix}\middle|2\right] = \frac{(-2n+1)}{(2\alpha+1)}\frac{(1/2)_{n-1}}{(\alpha+3/2)_{n-1}} \tag{4.26}$$

and using the known summation theorem ([19], Eq. (10), p. 127), we now get the presumed new theorem

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}-n, 3/2-n, \alpha/2, \alpha/2+1/2\\ 1/2, -\alpha/2-n+1/2, -\alpha/2-n+1\end{matrix}\middle| 1\right] \\
&= \frac{(2\alpha)_{2n}}{2(2n-1)(\alpha)_{2n}}\left\{(2n-1)\ \frac{(1/2)_n}{(\alpha+1/2)_n} - \frac{n(2n-1)}{(2\alpha+1))}\ \frac{(1/2)_{n-1}}{(\alpha+3/2)_{n-1}} + n - 1\right\}.
\end{aligned} \tag{4.27}
$$

Here is another way. Using the Euler transformation [19, Th. 20, p. 60] on the left side of (4.24), we obtain

$$
\begin{aligned}
&(1-z)^{\frac{3}{2}-\frac{a}{2}}{}_2F_1\left[\begin{matrix}1/2-a/2, a/2-3/2\\ 1/2\end{matrix}\middle| -\frac{z^2}{4(1-z)}\right] \\
&= \frac{1}{4(a-2)}\Big[(a-1)+(a-1)(1-z)^{3-a}+(a-3)(1-z)^{2-a}+(a-3)(1-z)\Big]
\end{aligned} \tag{4.28}
$$

and applying the operator ${}_zO^{\alpha}_{\beta}\{\}\big|_{z=1}$ on each side of the equation (4.28), we obtain after simplifications

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}1/2-a/2, a/2-3/2, \alpha/2, \alpha/2+1/2\\ 1/2, \beta-a/2+3/2, \alpha-\beta+a/2-1/2\end{matrix}\middle| 1\right] \\
&= \frac{\Gamma(\beta-\frac{a}{2}+\frac{3}{2})}{4(a-2)\Gamma(\beta-\alpha-\frac{a}{2}+\frac{3}{2})}\left\{\frac{\Gamma(\beta-\alpha)}{\Gamma(\beta+1)}(2(a-2)\beta-(a-3)\alpha)\right. \\
&\left. + \frac{\Gamma(\beta-\alpha+2-a)}{\Gamma(\beta+3-a)}(2(a-2)\beta-(a-1)\alpha-2(a-2)^2)\right\}
\end{aligned} \tag{4.29}
$$

a valid identity if: a) a is a positive odd integer or negative integer, b) α is a negative integer or a) and b) together.

Example 4.3. For each of the transformation formulas in section **5**, we can explore different ways to use this operator to discover known but sometimes new results. As an example, returning to equation (4.9), using (3.10) and the fact that

$$
(w+z-wz)^{\beta-\alpha-1} = \sum_{k=0}^{\infty}\frac{(1+\alpha-\beta)_k}{k!}w^{\beta-\alpha-1-k}z^k(w-1)^k,
$$

we have

$$
\begin{aligned}
&{}_{(\frac{z}{2-z})^2}O^{\alpha}_{\beta}f(z) = 2^{2\beta-2\alpha}(2-z)^{2\alpha} \\
&\sum_{k=0}^{\infty}\frac{(1+\alpha-\beta)_k}{k!}(-1)^k z^{-\alpha-k}(1-z)^k{}_zO^{\alpha}_{\beta}\Big\{z^{\alpha+k}(2-z)^{1-2\beta}f(z)\Big\} \\
&= 2^{2\beta-2\alpha}(2-z)^{2\alpha}\frac{\Gamma(\beta)\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\beta+\alpha)} \\
&\sum_{k=0}^{\infty}\frac{(1+\alpha-\beta)_k(2\alpha)_k}{(\beta+\alpha)_k k!}(-1)^k(1-z)^k{}_zO^{2\alpha+k}_{\beta+\alpha+k}\Big\{(2-z)^{1-2\beta}f(z)\Big\}.
\end{aligned} \tag{4.30}
$$

If we put $f(z) = (1-(z/(2-z))^2)^{-\gamma}$, by (3.10) and (3.15), we have

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}\alpha, \gamma\\ \beta\end{matrix}\middle| \left(\frac{z}{2-z}\right)^2\right] = 2^{2\beta-2\alpha-1}(2-z)^{2\alpha}\frac{\Gamma(\beta)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta+\alpha)} \\
&\sum_{k=0}^{\infty}\frac{(1+\alpha-\beta)_k(2\alpha)_k}{(\beta+\alpha)_k k!}(-1)^k(1-z)^k{}_zO^{2\alpha+k}_{\beta+\alpha+k}\Big\{(2-z)^{1-2\beta+2\gamma}2^{1-2\beta}(1-z)^{-\gamma}\Big\}.
\end{aligned} \tag{4.31}
$$

Using (4.19), the last equation becomes after simplifications

$$
{}_2F_1\left[\begin{matrix}\alpha, \gamma\\ \beta\end{matrix}\middle| \left(\frac{z}{2-z}\right)^2\right] = (1-z/2)^{2\alpha}\frac{\Gamma(\beta)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta+\alpha)}\sum_{k=0}^{\infty}\frac{(1+\alpha-\beta)_k(2\alpha)_k}{(\beta+\alpha)_k k!} \tag{4.32}
$$

$$(-1)^k(1-z)^k F_1(2\alpha+k; 2\beta-2\gamma-1,\gamma;\alpha+\beta+k; z/2, z).$$

If $\beta = \alpha + 1$, we find again (4.21) and with $\beta = \alpha + 2$, we have the reduced formula

$$(1-z/2)^{-2\alpha}\ {}_2F_1\left[\begin{matrix}\alpha,\gamma\\ \alpha+2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] = \frac{\alpha+1}{2\alpha+1}F_1(2\alpha; 2\alpha-2\gamma+3,\gamma; 2\alpha+2; z/2, z) \tag{4.33}$$
$$+\frac{\alpha}{2\alpha+1}(1-z)F_1(2\alpha+1; 2\alpha-2\gamma+3,\gamma; 2\alpha+3; z/2, z).$$

In (4.32), if $z = 0$, we obtain Kummer's theorem [23, Eq. (III-5), p. 243]. With $z = 1$, using the first Gauss summation theorem (4.17) and the Legendre duplication formula $\Gamma(1/2)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)$, we obtain

$$F_1\left(2\alpha; 2\beta-2\gamma-1,\gamma;\alpha+\beta;\frac{1}{2},1\right) = \frac{\Gamma(\beta+\alpha)\Gamma(\beta-\alpha-\gamma)}{\Gamma(\beta-\alpha)\Gamma(\beta+\alpha-\gamma)}\ {}_2F_1\left[\begin{matrix}2\alpha, 2\beta-2\gamma-1\\ \alpha+\beta-\gamma\end{matrix}\middle|\frac{1}{2}\right] \tag{4.34}$$
$$= \frac{\sqrt{\pi}\ \Gamma(\beta+\alpha)\Gamma(\beta-\alpha-\gamma)}{\Gamma(\alpha+1/2)\Gamma(\beta-\alpha)\Gamma(\beta-\gamma)}$$

which is the second Gauss summation theorem [23, Eq. (III-6), p. 243].

Example 4.4. Returning to equation (4.30), using transformation (4.7) and properties (3.10) and (3.15), we obtain

$${}_3F_2\left[\begin{matrix}\alpha, a/2, a/2+1/2\\ \beta, b+1/2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] = (1-z/2)^{2\alpha}\frac{\Gamma(\beta)\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta+\alpha)} \tag{4.35}$$
$$\sum_{k=0}^{\infty}\frac{(1+\alpha-\beta)_k(2\alpha)_k}{(\beta+\alpha)_k k!}(-1)^k(1-z)^k\, {}_zO^{2\alpha+k}_{\beta+\alpha+k}\left\{(1-z/2)^{1-2\beta+a}\,{}_2F_1\left[\begin{matrix}a,b\\ 2b\end{matrix}\middle|z\right]\right\}.$$

With $\beta = a/2 + 1/2$, (4.35) reduces to

$${}_2F_1\left[\begin{matrix}\alpha, a/2\\ b+1/2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] = (1-z/2)^{2\alpha}\frac{\Gamma(1+2\alpha)\Gamma(1/2+a/2)}{\Gamma(1+\alpha)\Gamma(1/2+a/2+\alpha)} \tag{4.36}$$
$$\sum_{k=0}^{\infty}\frac{(1/2-a/2+\alpha)_k(2\alpha)_k}{(1/2+a/2+\alpha)_k k!}(z-1)^k\ {}_3F_2\left[\begin{matrix}2\alpha+k, a, b\\ 1/2+a/2+\alpha+k, 2b\end{matrix}\middle|z\right].$$

Putting $z = 1$ in (4.36) and using (4.17), we obtain

$${}_3F_2\left[\begin{matrix}2\alpha, a, b\\ \alpha+a/2+1/2, 2b\end{matrix}\middle|1\right] \tag{4.37}$$
$$= \frac{\sqrt{\pi}\ \Gamma(1/2+b)\Gamma(1/2+a/2+\alpha)\Gamma(1/2+b-a/2-\alpha)}{\Gamma(1/2+\alpha)\Gamma(1/2+a/2)\Gamma(1/2+b-a/2)\Gamma(1/2+b-\alpha)}$$

which is the Watson summation theorem [23, Eq. (III-23), p. 245]. If $z = 0$ in (4.35), we obtain the Kummer summation theorem [23, Eq. (III-5), p. 243]

$${}_2F_1\left[\begin{matrix}2\alpha, 1+\alpha-\beta\\ \alpha+\beta\end{matrix}\middle|-1\right] = \frac{\Gamma(1+\alpha)\Gamma(\alpha+\beta)}{\Gamma(1+2\alpha)\Gamma(\beta)}. \tag{4.38}$$

If $\beta = \alpha$ in (4.35), we find (4.7).

Example 4.5. If we choose $g(z) = z/(2-z)$ in (2.5), with the same properties, we can get

$${}_{\frac{z}{2-z}}O^{\theta}_{\delta}f(z) = 2^{\delta-\theta}(2-z)^{\theta}\,{}_zO^{\theta}_{\delta}(2-z)^{-\delta}f(z) \tag{4.39}$$

and, with (4.7), we easily obtain

$$
{}_4F_3\left[\begin{matrix}\theta/2, \theta/2+1/2, \alpha/2, \alpha/2+1/2\\ \delta/2, \delta/2+1/2, b+1/2\end{matrix}\middle|(\frac{z}{2-z})^2\right] = \tag{4.40}
$$
$$
2^{\delta-\theta-\alpha}(2-z)^{\theta}{}_zO_{\delta}^{\theta}(2-z)^{\alpha-\delta}{}_2F_1\left[\begin{matrix}\alpha, b\\ 2b\end{matrix}\middle|z\right].
$$

If $b=\alpha/2$, with (4.19), (4.40) becomes

$$
{}_3F_2\left[\begin{matrix}\theta/2, \theta/2+1/2, \alpha/2\\ \delta/2, \delta/2+1/2\end{matrix}\middle|(\frac{z}{2-z})^2\right] = (1-z/2)^{\theta}F_1(\theta;\delta-\alpha,\alpha/2;\delta;z/2,z). \tag{4.41}
$$

Putting $z=1$ in (4.41), we obtain

$$
{}_3F_2\left[\begin{matrix}\theta/2, \theta/2+1/2, \alpha/2\\ \delta/2, \delta/2+1/2\end{matrix}\middle|1\right] = 2^{-\theta}F_1(\theta;\delta-\alpha,\alpha/2;\delta;1/2,1) \tag{4.42}
$$
$$
= 2^{-\theta}\frac{\Gamma(\delta)\Gamma(\delta-\theta-\alpha/2)}{\Gamma(\delta-\theta)\Gamma(\delta-\alpha/2)}{}_2F_1\left[\begin{matrix}\theta, \delta-\alpha\\ \delta-\alpha/2\end{matrix}\middle|\frac{1}{2}\right].
$$

With $\theta = 1+\alpha-\delta$ in (4.42), using the Bailey summation theorem [23, Eq. (III-7), p. 243], we obtain after simplifications

$$
{}_3F_2\left[\begin{matrix}1/2+\alpha/2-\delta/2, 1+\alpha/2-\delta/2, \alpha/2\\ \delta/2, \delta/2+1/2\end{matrix}\middle|1\right] = \tag{4.43}
$$
$$
2^{-\alpha/2}\frac{\sqrt{\pi}\,\Gamma(\delta)\Gamma(2\delta-3\alpha/2-1)}{\Gamma(2\delta-\alpha-1)\Gamma(\delta-3\alpha/4)\Gamma(1/2+\alpha/4)}
$$

which is a special case of the Whipple summation theorem ([19, Eq.(1), p. 90].

These four examples show how special functions can be explored using the operator ${}_{g(z)}O_{\beta}^{\alpha}$ and its properties. This could potentially be done for each of the transformation formulas in the next section.

5. List of some transformation formulas involving hypergeometric function

In this section, we give a long list of presumed new formulas for transforming hypergeometric functions with quadratic arguments. We limit our study here to arguments $g(z)$ equal z^2, $4z(1-z)$, $z/(2-z)$, $(\frac{z}{2-z})^2$, $-\frac{4z}{(1-z)^2}$ and $-\frac{z^2}{4(1-z)}$. In a future article, we will treat the case of transformations with cubic arguments and higher degrees. The method for obtaining them is illustrated with the help of several cases in [26]. It is based on the use of the generalized chain rule (2.5) and several properties of the well-formed fractional calculator ${}_{g(z)}O_{\beta}^{\alpha}$ in section **3** (more precisely, formulas (3.1), (3.4), (3.10), (3.11), (3.12) and (3.15)) applied in particular to several Gaussian transformation formulas appearing in the list given in 1881 by Goursat [6] and Rainville [19]. In principle, the proposed method can be applied to all transformations of hypergeometric functions. The purpose of this article is also to increase the efficiency of the operator ${}_{g(z)}O_{\beta}^{\alpha}$ in discovering new formulas involving special functions.

It would be possible to apply the method for any values of α and β (except $\beta = 0, -1, -2, \ldots$) using for example the Bell polynomials. The general case will also be considered in a future paper. For the moment, we limit our study practically only to the cases $\beta - \alpha = n$ with $n = 1, 2$ because the number of terms increases considerably when n is very large. It is important to note that the method can be applied also recursively to all formulas obtained from the list. For example: the transformation (5.6) is obtained from (4.12) with (5.1); (5.39) and (5.45) are obtained from (5.36) respectively with (5.1) and (5.2); and transformation (5.60) is obtained from (5.56) with (5.1).

With $\beta = \alpha + 1$, the relation (2.5) is reduced to

$$ {}_{g(z)}O^{\alpha}_{\alpha+1} f(z) = \Big(\frac{g(z)}{h(z)}\Big)^{-\alpha} \Big\{ {}_{h(z)}O^{\alpha}_{\alpha+1} \Big(\frac{g(z)}{h(z)}\Big)^{\alpha-1} \frac{g'(z)}{h'(z)} f(z) \Big\} \tag{5.1} $$

and with $\beta = \alpha + 2$, relation (2.5) becomes

$$ \begin{aligned} &{}_{g(z)}O^{\alpha}_{\alpha+2} f(z) \\ &= \Big(\frac{g(z)}{h(z)}\Big)^{-\alpha-1} \Big\{ {}_{h(z)}O^{\alpha}_{\alpha+2} \Big(\frac{g(z)}{h(z)}\Big)^{\alpha-1} \frac{g'(z)}{h'(z)} \Big(\frac{g(z)-g(w)}{h(z)-h(w)}\Big) f(z) \Big\}\Big|_{w=z}. \end{aligned} \tag{5.2} $$

For each new transformation obtained in the list, we indicate the reference of the initial transformation $f(z)$ and relevant rules used and conditions imposed. Some special cases of these new transformations are also given.

Case 5.1. *((4.12); (5.1); $g(z) = z/(2-z)$, $h(z) = z$)*

$$ \begin{aligned} &2^{a-3}(2-z)^{4-a}\, {}_3F_2\left[\begin{matrix} a/2-2, a/2, a/2-3/2 \\ a/2-1, b+1/2 \end{matrix}\Big| \Big(\frac{z}{2-z}\Big)^2\right] \\ &= \frac{(a-3)}{(a-2)}\, {}_3F_2\left[\begin{matrix} a-4, a-2, b \\ a-3, 2b \end{matrix}\Big| z\right] + \frac{(a-1)}{(a-2)}\, {}_3F_2\left[\begin{matrix} a-4, a, b \\ a-1, 2b \end{matrix}\Big| z\right] \\ &- \frac{(a-4)}{(a-2)} z\, {}_2F_1\left[\begin{matrix} a-3, b \\ 2b \end{matrix}\Big| z\right]. \end{aligned} \tag{5.3} $$

Using the contiguous relations (3.17),the transformation (5.3) becomes after simplifications

$$ \begin{aligned} &2^{a-3}(2-z)^{4-a}\, {}_3F_2\left[\begin{matrix} a/2-2, a/2, a/2-3/2 \\ a/2-1, b+1/2 \end{matrix}\Big| \Big(\frac{z}{2-z}\Big)^2\right] \\ &= (a-2)^{-1}\Big\{ 2(a-4)(2-z)\, {}_2F_1\left[\begin{matrix} a-3, b \\ 2b \end{matrix}\Big| z\right] + 4\, {}_2F_1\left[\begin{matrix} a-4, b \\ 2b \end{matrix}\Big| z\right]\Big\}. \end{aligned} \tag{5.4} $$

If $z = 1$ in (5.4), we find the following summation theorem

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a/2, a/2-2, a/2-3/2 \\ a/2-1, b+1/2 \end{matrix}\Big| 1\right] \\ &= \frac{2^{2b-a+2}\Gamma(b+1/2)\Gamma(3+b-a)(2ab-4b+3a-a^2)}{\sqrt{\pi}\,(a-2)\Gamma(4+2b-a)}. \end{aligned} \tag{5.5} $$

Case 5.2. *([6, Eq. (45), p. 120];(2.5); $g(z) = z/(2-z)$, $h(z) = z$, $a = 2b$)*

$$ F_1(\alpha;\beta-2b, b;\beta; z/2, z) = (1-z/2)^{-\alpha}\, {}_3F_2\left[\begin{matrix} \alpha/2, \alpha/2+1/2, b \\ \beta/2, \beta/2+1/2 \end{matrix}\Big| \Big(\frac{z}{2-z}\Big)^2\right]. \tag{5.6} $$

If $z = 1$ in (5.6), we have the transformation formula

$$ {}_3F_2\left[\begin{matrix} \alpha/2, \alpha/2+1/2, b \\ \beta/2, \beta/2+1/2 \end{matrix}\Big| 1\right] = \frac{2^{-\alpha}\Gamma(\beta)\Gamma(\beta-\alpha-b)}{\Gamma(\beta-\alpha)\Gamma(\beta-b)}\, {}_2F_1\left[\begin{matrix} \alpha, \beta-2b \\ \beta-b \end{matrix}\Big| \frac{1}{2}\right]. \tag{5.7} $$

If $\beta = 1 + \alpha$ in (5.7), we find the second Gauss theorem [23, Eq. (III-6), p. 243]. With $\beta = 1 + 2b - \alpha$, we obtain a special case of the Dixon summation theorem [19, Th. 33, p. 92]

$$ {}_3F_2\left[\begin{matrix} b, \alpha/2, \alpha/2+1/2 \\ 1-\alpha/2+b, 1/2-\alpha/2+b \end{matrix}\Big| 1\right] = \frac{2^{-2\alpha}\Gamma(1/2-\alpha+b/2)\Gamma(1-\alpha+2b)}{\Gamma(1/2-2\alpha+2b)\Gamma(1/2+b/2)}. \tag{5.8} $$

Case 5.3. *([6, Eq. (45), p. 120]; (5.1); $g(z) = z/(2-z)$, $h(z) = z/(1-z)$)*

$$\Big(\frac{1-z}{1-z/2}\Big)^{a-1} {}_2F_1\left[\begin{matrix} a/2, a/2-1/2 \\ b+1/2 \end{matrix}\middle| \Big(\frac{z}{2-z}\Big)^2\right] = \sum_{n=0}^{\infty} \frac{(a-b)_n (a-1)_n}{n!(a)_n} \Big(\frac{-z}{1-z}\Big)^n {}_3F_2\left[\begin{matrix} a-1+n, 2b-a, b \\ a+n, 2b \end{matrix}\middle| \frac{-z}{1-z}\right]. \tag{5.9}$$

If $b = a/2$ in (5.9), with the Euler transformation [19, Eq. (4), p. 60], we obtain

$$(1-z)^{1/2-a/2} {}_2F_1\left[\begin{matrix} 1/2, a/2-1/2 \\ a/2+1/2 \end{matrix}\middle| \frac{-z^2}{4(1-z)}\right] = {}_2F_1\left[\begin{matrix} a/2, a-1 \\ a \end{matrix}\middle| z\right]. \tag{5.10}$$

Applying the operator ${}_zO_\beta^{-n}\{\}\big|_{z=1}$ on each side of the equation (5.10), we obtain after simplifications

$$ {}_4F_3\left[\begin{matrix} 1/2, a/2-1/2, -n/2, -n/2+1/2 \\ a/2+1/2, a/2+1/2-n-\beta, \beta-a/2+1/2 \end{matrix}\middle| 1\right] = \frac{(\beta)_n}{(\beta-a/2+1/2)_n} {}_3F_2\left[\begin{matrix} a/2, a-1, -n \\ a, \beta \end{matrix}\middle| 1\right] \tag{5.11}$$

If we put respectively $\beta = a/2$ and $\beta = a-1$, we obtain the following two summation theorems

$$ {}_3F_2\left[\begin{matrix} a/2-1/2, -n/2, -n/2+1/2 \\ -n+1/2, a/2+1/2 \end{matrix}\middle| 1\right] = \frac{n!(a/2)_n}{(a)_n (1/2)_n} \tag{5.12}$$

and

$$ {}_3F_2\left[\begin{matrix} 1/2, -n/2, -n/2+1/2 \\ a/2+1/2, 3/2-a/2-n \end{matrix}\middle| 1\right] = \frac{(a-1)(a/2)_n}{(a-1+n)(a/2-1/2)_n}. \tag{5.13}$$

If $a \to -n$ in (5.11), we obtain

$$ {}_3F_2\left[\begin{matrix} 1/2, -n/2, -n/2-1/2 \\ \beta+n/2+1/2, 1/2-3n/2-\beta \end{matrix}\middle| 1\right] = \frac{\Gamma(\beta+n/2+1/2)\Gamma(\beta+3n/2+1)}{(\beta+n)\Gamma(\beta+3n/2+1/2)\Gamma(\beta+n/2)}. \tag{5.14}$$

Note that the series in (5.11) must be terminated (with $\alpha = -n$ in ${}_zO_\beta^\alpha$, see (3.18)) because the point $z = 1$ is outside the convergence region $|\frac{-z^2}{4(1-z)}| < 1$ of the hypergeometric function ${}_2F_1$ in (5.10).

Case 5.4. *($f(z) = (1-z)^{-\gamma}(1-z/2)^{2\gamma}$; (2.5) with $\beta = 1+\alpha+n$; $g(z) = z/(2-z)$, $h(z) = z$)*

$$z^n (1-z/2)^{-2\alpha} {}_2F_1\left[\begin{matrix} \alpha, \gamma \\ \alpha+n+1 \end{matrix}\middle| \Big(\frac{z}{2-z}\Big)^2\right] = \frac{(1+\alpha)_n}{(1+2\alpha)_n} \sum_{k=0}^{n} \binom{n}{k} (z-1)^k F_1(2\alpha; 1-2\gamma+2\alpha+2n, \gamma-k; 2\alpha+n+1; z/2, z). \tag{5.15}$$

If $n = 0$ in (5.15), we obtain (4.21). If $n = 1$ in (5.15), we obtain

$$\frac{(1+2\alpha)}{(1+\alpha)} z(1-z/2)^{-2\alpha} {}_2F_1\left[\begin{matrix} \alpha, \gamma \\ \alpha+2 \end{matrix}\middle| \Big(\frac{z}{2-z}\Big)^2\right] = F_1(2\alpha; 3-2\gamma+2\alpha, \gamma; 2\alpha+2; z/2, z) - (1-z)F_1(2\alpha; 3-2\gamma+2\alpha, \gamma-1; 2\alpha+2; z/2, z). \tag{5.16}$$

Case 5.5. *($f(z) = (1-z)^{-\gamma}(1-z/2)^{2\gamma}$; (2.5) with $\beta = \alpha + n + 1$; $g(z) = z^2/(2-z)^2$, $h(z) = z$)*

$$ {}_2F_1\left[\begin{matrix}\alpha,\gamma\\ \alpha+n+1\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] = \frac{(\alpha+1)_n(2\alpha)_n}{(1+2\alpha)_{2n}}(1-z/2)^{2\alpha} \sum_{k=0}^{n}\binom{n}{k}\frac{(-2\alpha-2n)_k}{(1-2\alpha-n)_k} F_1(2\alpha+n-k;1-2\gamma+2\alpha+2n,\gamma-k;2\alpha+2n-k+1;z/2,z). \tag{5.17} $$

If $n = 1$ in (5.17), we obtain

$$ (2\alpha+1)(1-z/2)^{-2\alpha}{}_2F_1\left[\begin{matrix}\alpha,\gamma\\ \alpha+2\end{matrix}\middle|\left(\frac{z}{2-z}\right)^2\right] = \alpha F_1(2\alpha+1;3-2\gamma+2\alpha,\gamma;2\alpha+3;z/2,z) + (\alpha+1)F_1(2\alpha;3-2\gamma+2\alpha,\gamma-1;2\alpha+2;z/2,z). \tag{5.18} $$

Case 5.6. *([19, Th. 25, p. 67]; (5.1); $g(z) = z(1-z)$, $h(z) = z$)*

$$ (1-z)^{a+b+1/2}{}_2F_1\left[\begin{matrix}a+1/2,b+1/2\\ a+b+3/2\end{matrix}\middle|4z(1-z)\right] = {}_2F_1\left[\begin{matrix}1/2-a+b,1/2+a-b\\ a+b+3/2\end{matrix}\middle|z\right]. \tag{5.19} $$

which is equivalent to [19, Th. 25, p. 67].

If $z = 1/2$ in (5.19), using the Gauss summation theorem [19, Th. 18, p. 49], we find

$$ {}_2F_1\left[\begin{matrix}1/2-a+b,1/2+a-b\\ a/2+b/2+3/2\end{matrix}\middle|\frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(a+b+3/2)}{2^{a+b+1/2}\Gamma(1+a)\Gamma(1+b)} \tag{5.20} $$

which is a special case of the Bailey theorem [23, Th. 27, p. 69], if we use [19, Th. 27, p. 69].
If $b = a - 1/2$ in (5.19), we have

$$ {}_2F_1\left[\begin{matrix}a,a+1/2\\ 2a+1\end{matrix}\middle|4z(1-z)\right] = (1-z)^{-2a}, \tag{5.21} $$

and applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on the both sides of (5.21), we obtain

$$ {}_4F_3\left[\begin{matrix}a,a+1/2,-n,\beta+2a+n\\ 2a+1,\beta/2+a,\beta/2+a+1/2\end{matrix}\middle|1\right] = \frac{(\beta)_n}{(\beta+2a)_n} \tag{5.22} $$

which seems new.

Case 5.7. *([19, Th. 25, p. 67]; (5.2); $g(z) = z(1-z)$, $h(z) = z$)*

$$ (1-z)^{a+b+3/2}{}_2F_1\left[\begin{matrix}a+1/2,b+1/2\\ a+b+5/2\end{matrix}\middle|4z(1-z)\right] = (1-z){}_2F_1\left[\begin{matrix}1/2-a+b,1/2+a-b\\ a+b+5/2\end{matrix}\middle|z\right] - \frac{(2a+2b+1)}{(2a+2b+5)}z\;{}_3F_2\left[\begin{matrix}3/2+a+b,1/2-a+b,1/2+a-b\\ 7/2+a+b,a+b+1/2\end{matrix}\middle|z\right]. \tag{5.23} $$

If $z = 1/2$ in (5.23), using the Gauss summation theorem [19, Th. 18, p. 49] and the Bailey theorem [23, Eq. (III-7), p. 243], we obtain

$$ {}_3F_2\left[\begin{matrix}1/2-a+b,1/2+a-b,3/2+a+b\\ a+b+1/2,a+b+7/2\end{matrix}\middle|\frac{1}{2}\right] \tag{5.24} $$

$$= \frac{\sqrt{\pi}\,\Gamma(a+b+7/2)\Big\{\Gamma(a+2)\Gamma(b+2)-\Gamma(a+3/2)\Gamma(b+3/2)\Big\}}{2^{a+b+1/2}(2a+2b+1)\Gamma(a+2)\Gamma(b+2)\Gamma(a+3/2)\Gamma(b+3/2)}.$$

If $b = a - 1/2$ in (5.23), the equation becomes

$$(1-z)^{2a+1}{}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+2\end{matrix}\middle| 4z(1-z)\right] = 1 - \frac{2a+1}{a+1}z. \tag{5.25}$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.25), we obtain after simplifications the next summation theorem

$${}_4F_3\left[\begin{matrix}a, a+1/2, -n, \beta+n\\ 2a+2, \beta/2, \beta/2+1/2\end{matrix}\middle| 1\right] = \frac{(\beta-2a-1)_n}{(\beta)_n}\Big\{1 + \frac{(2a+1)n}{(\beta-2a-1)(a+1)}\Big\}. \tag{5.26}$$

Case 5.8. *((5.19); (5.1); $g(z) = z(1-z)$, $h(z) = z$)*

$$\begin{aligned}(1-z)^{a+b+3/2}{}_2F_1\left[\begin{matrix}a+1/2, b+1/2\\ a+b+5/2\end{matrix}\middle| 4z(1-z)\right] &= {}_2F_1\left[\begin{matrix}1/2-a+b, 1/2+a-b\\ a+b+5/2\end{matrix}\middle| z\right] \\ -2\frac{(2a+2b+3)}{(2a+2b+5)}z\;{}_3F_2&\left[\begin{matrix}5/2+a+b, 1/2-a+b, 1/2+a-b\\ 7/2+a+b, a+b+3/2\end{matrix}\middle| z\right].\end{aligned} \tag{5.27}$$

If $z = 1/2$ in (5.27), using the Gauss summation theorem [19, Th. 18, p. 49] and the Bailey theorem [23, Eq. (III-7), p. 243], we obtain

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}1/2-a+b, 1/2+a-b, 5/2+a+b\\ a+b+3/2, a+b+7/2\end{matrix}\middle| \frac{1}{2}\right] \\ &= \frac{\sqrt{\pi}\,\Gamma(a+b+7/2)\Big\{2\Gamma(a+2)\Gamma(b+2)-\Gamma(a+3/2)\Gamma(b+3/2)\Big\}}{2^{a+b+3/2}(2a+2b+3)\Gamma(a+2)\Gamma(b+2)\Gamma(a+3/2)\Gamma(b+3/2)}.\end{aligned} \tag{5.28}$$

If $b = a + 1/2$ in (5.27), we obtain

$$(1-z)^{2a+2}{}_2F_1\left[\begin{matrix}a+1, a+1/2\\ 2a+3\end{matrix}\middle| 4z(1-z)\right] = 1 - 4\frac{(a+1)}{(2a+3)}z. \tag{5.29}$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.29), we obtain

$${}_4F_3\left[\begin{matrix}a+1, a+1/2, -n, \beta+2a+2+n\\ 2a+3, \beta/2+a+1, \beta/2+a+3/2\end{matrix}\middle| 1\right] = \frac{(\beta)_n}{(\beta+2a+2)_n}\Big\{1 + 4\frac{(a+1)n}{(2a+3)\beta}\Big\}. \tag{5.30}$$

With (5.27) and (5.23), we deduce the relation

$$\begin{aligned}&(2a+2b+5)\;{}_2F_1\left[\begin{matrix}1/2-a+b, 1/2+a-b\\ a+b+5/2\end{matrix}\middle| z\right] \\ &= 2(2a+2b+3)\;{}_3F_2\left[\begin{matrix}5/2+a+b, 1/2-a+b, 1/2+a-b\\ 7/2+a+b, a+b+3/2\end{matrix}\middle| z\right] \\ &\quad - (2a+2b+1)\;{}_3F_2\left[\begin{matrix}3/2+a+b, 1/2-a+b, 1/2+a-b\\ 7/2+a+b, a+b+1/2\end{matrix}\middle| z\right].\end{aligned} \tag{5.31}$$

Case 5.9. *((5.19); (5.2); $g(z) = z(1-z)$, $h(z) = z$)*

$$(1-z)^{a+b+3/2}{}_2F_1\left[\begin{matrix}a,b\\a+b+5/2\end{matrix}\middle|4z(1-z)\right] = (1-z){}_2F_1\left[\begin{matrix}1/2-a+b,1/2+a-b\\a+b+5/2\end{matrix}\middle|z\right]$$
$$-z(3-2z)\frac{(2a+2b+1)}{(2a+2b+5)}\,{}_3F_2\left[\begin{matrix}3/2+a+b,1/2-a+b,1/2+a-b\\7/2+a+b,a+b+1/2\end{matrix}\middle|z\right]$$
$$+2z^2\frac{(2a+2b+1)(2a+2b+3)}{(2a+2b+5)(2a+2b+7)}{}_3F_2\left[\begin{matrix}5/2+a+b,1/2-a+b,1/2+a-b\\9/2+a+b,a+b+1/2\end{matrix}\middle|z\right]. \tag{5.32}$$

If $z = 1/2$ in (5.32), using the Bailey theorem [23, Eq. (III-7), p. 243] and (5.28), we obtain after simplifications

$$ {}_3F_2\left[\begin{matrix}1/2-a+b,1/2+a-b,5/2+a+b\\a/2+b/2+9/2,a+b+1/2\end{matrix}\middle|\frac{1}{2}\right]$$
$$=\frac{(2a+2b+5)(2a+2b+7)}{(2a+2b+1)(2a+2b+3)}\frac{\sqrt{\pi}\,\Gamma(a+b+5/2)}{2^{a+b+7/2}\Gamma(a+5/2)\Gamma(b+5/2)\Gamma(a+2)\Gamma(b+2)}$$
$$\Big\{(4ab+6a+6b+15)\Gamma(a+2)\Gamma(b+2)-8\Gamma(a+5/2)\Gamma(b+5/2)\Big\}. \tag{5.33}$$

If $b = a - 1/2$ in (5.32), we have

$$(1-z)^{2a+1}{}_2F_1\left[\begin{matrix}a-1/2,a\\2a+2\end{matrix}\middle|4z(1-z)\right] = 1-\frac{(4a+1)}{(a+1}z+\frac{8a}{(2a+3)}z^2, \tag{5.34}$$

and applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.34), we obtain

$$ {}_4F_3\left[\begin{matrix}a-1/2,a,-n,\beta+2a+1+n\\2a+2,\beta/2+a+1/2,\beta/2+a+1\end{matrix}\middle|1\right]$$
$$=\frac{(\beta)_n}{(\beta+2a+1)_n}\Big\{1+\frac{(4a+1)n}{(a+1)\beta}+\frac{8an(n-1)}{(2a+3)\beta(\beta+1)}\Big\}. \tag{5.35}$$

Case 5.10. *([19, Th. 25, p. 67]; (5.2); $g(z) = z(1-z)$, $h(z) = z/(1-z)$)*

$$(1-z)^{2a}{}_3F_2\left[\begin{matrix}a+1/2,a-1,b+1/2\\a+1,a+b+1/2\end{matrix}\middle|4z(1-z)\right]$$
$$=(1-z)^2{}_3F_2\left[\begin{matrix}2a,a-1,1/2+a-b\\a+1,a+b+1/2\end{matrix}\middle|-\frac{z}{1-z}\right]$$
$$-\frac{(a-1)}{(a+1)}z^2{}_3F_2\left[\begin{matrix}2a,a,1/2+a-b\\a+2,a+b+1/2\end{matrix}\middle|-\frac{z}{1-z}\right]. \tag{5.36}$$

If $b = a + 1/2$ in (5.36), we obtain as a special case

$$ {}_2F_1\left[\begin{matrix}a+1/2,a-1\\2a+1\end{matrix}\middle|4z(1-z)\right] = (1-z)^{2-2a}-\frac{(a-1)}{(a+1)}z^2(1-z)^{-2a}. \tag{5.37}$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.36), we obtain after simplifications the next summation theorem

$$ {}_4F_3\left[\begin{matrix}a+1/2,a-1,-n,\beta+n\\2a+1,\beta/2,\beta/2+1/2\end{matrix}\middle|1\right]$$
$$=\frac{(\beta-2a+2)_n}{(\beta)_n}\Big\{1-\frac{(a-1)n(n-1)}{(\beta+n-2a+1)(\beta+n-2a)(a+1)}\Big\}. \tag{5.38}$$

Case 5.11. *([19, Th. 27, p. 69]; (5.1); $g(z) = z(1-z)$, $h(z) = z$)*

$$(1-z)^c {}_2F_1\left[\begin{matrix} c/2-a/2, c/2+a/2-1/2 \\ c+1 \end{matrix}\middle| 4z(1-z)\right] = {}_2F_1\left[\begin{matrix} a, 1-a \\ c+1 \end{matrix}\middle| z\right] - 2z\frac{c}{c+1}{}_3F_2\left[\begin{matrix} a, 1-a, c+1 \\ c, c+2 \end{matrix}\middle| z\right]. \tag{5.39}$$

If $z = 1/2$ in (5.39), we obtain

$$ {}_3F_2\left[\begin{matrix} a, 1-a, c+1 \\ c, c+2 \end{matrix}\middle| \frac{1}{2}\right] = 2^{-c-1}\frac{\Gamma(c+2)\sqrt{\pi}}{c} \frac{(2\Gamma(c/2+a/2+1)\Gamma(c/2-a/2+3/2) - \Gamma(c/2+a/2+1/2)\Gamma(c/2-a/2+1))}{\Gamma(c/2+a/2+1/2)\Gamma(c/2-a/2+1)\Gamma(c/2+a/2+1)\Gamma(c/2-a/2+3/2)} \tag{5.40}$$

which is equivalent to the summation theorem found in [26, Eq. (4.12)].

In addition, applying the same procedure as that used in Case 5.7, we find results equivalent to (5.25) and (5.26). Moreover, using the contiguous relation (3.17), we obtain a summation theorem of the same type as the Bailey theorem [23, Eq. (III-7), p. 243]

$$ {}_2F_1\left[\begin{matrix} a, 3-a \\ c \end{matrix}\middle| \frac{1}{2}\right] = \frac{\sqrt{\pi}\,2^{-c+3}\Gamma(c)}{(a-1)(a-2)} \left\{\frac{c-2}{\Gamma(c/2+a/2-1)\Gamma(c/2-a/2+1/2)} - \frac{2}{\Gamma(c/2+a/2-3/2)\Gamma(c/2-a/2)}\right\}. \tag{5.41}$$

Case 5.12. *((5.36); (5.1); $g(z) = z(1-z)$, $h(z) = z$)*

$$\begin{aligned}(1-z)^{c+1} {}_2F_1\left[\begin{matrix} c/2-a/2, c/2+a/2-1/2 \\ c+2 \end{matrix}\middle| 4z(1-z)\right] &= {}_2F_1\left[\begin{matrix} a, 1-a \\ c+2 \end{matrix}\middle| z\right] - 2\frac{c}{c+2}z\,{}_3F_2\left[\begin{matrix} a, 1-a, c+1 \\ c, c+3 \end{matrix}\middle| z\right] \\ &- 2\frac{c+1}{c+2}z\,{}_3F_2\left[\begin{matrix} a, 1-a, c+2 \\ c+1, c+3 \end{matrix}\middle| z\right] + 4z^2\frac{c}{(c+3)}{}_4F_3\left[\begin{matrix} a, 1-a, c+1, c+3 \\ c, c+2, c+4 \end{matrix}\middle| z\right].\end{aligned} \tag{5.42}$$

If $a = 1-c$ in (5.42), we obtain as a special case

$$(1-z)^{c+1} {}_2F_1\left[\begin{matrix} c/2-1/2, c/2 \\ c+2 \end{matrix}\middle| 4z(1-z)\right] = \{1 - \frac{2(2c-1)}{(c+2)}z + \frac{4c}{(c+3)}z^2\}. \tag{5.43}$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.43), we obtain after simplifications the next summation theorem

$$ {}_4F_3\left[\begin{matrix} c/2-1/2, c/2, -n, \beta+c+1+n \\ c+2, \beta/2+c/2+1/2, \beta/2+c/2+1 \end{matrix}\middle| 1\right] = \frac{(\beta)_n}{(\beta+c+1)_n}\left\{1 + \frac{2(2c-1)n}{(c+1)\beta} + \frac{4cn(n-1)}{(c+3)\beta(\beta+1)}\right\}. \tag{5.44}$$

Case 5.13. *([19, Th. 27, p. 69]; (5.2); $g(z) = z(1-z)$, $h(z) = z$)*

$$(1-z)^{c+1} {}_2F_1\left[\begin{matrix} c/2-a/2, c/2+a/2-1/2 \\ c+2 \end{matrix}\middle| 4z(1-z)\right] \tag{5.45}$$

$$= (1-z)_2F_1\left[\begin{matrix}a, 1-a\\ c+2\end{matrix}\middle| z\right] + z(2z-3)\frac{c}{c+2}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+1\\ c, c+3\end{matrix}\middle| z\right]$$
$$+ 2z^2\frac{c(c+1)}{(c+2)(c+3)}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+2\\ c, c+4\end{matrix}\middle| z\right].$$

If $z = 1/2$ in (5.45), with the help of (5.37),(3.17) and the Bailey theorem [23, Eq. (III-7), p. 243], we can deduce the following summation theorem

$${}_3F_2\left[\begin{matrix}a, 1-a, c+2\\ c, c+4\end{matrix}\middle| 1\right] \tag{5.46}$$
$$= \sqrt{\pi}\,2^{c-3}(c+2)(c+3)\Big\{\frac{(a^2-c^2-a-5c)}{\Gamma(c/2-a/2+5/2)\Gamma(c/2+a/2+2)}$$
$$+ \frac{8}{\Gamma(c/2-a/2+3/2)\Gamma(c/2+a/2+1)} - \frac{8}{\Gamma(c/2-a/2+2)\Gamma(c/2+a/2+3/2)}\Big\}.$$

Case 5.14. *((5.36); (5.2); $g(z) = z(1-z)$, $h(z) = z$)*

$$(1-z)^{c+2}{}_2F_1\left[\begin{matrix}c/2-a/2, c/2+a/2-1/2\\ c+3\end{matrix}\middle| 4z(1-z)\right] \tag{5.47}$$
$$= (1-z)_2F_1\left[\begin{matrix}a, 1-a\\ c+3\end{matrix}\middle| z\right] - 2z(1-z)\frac{c}{(c+3)}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+1\\ c, c+4\end{matrix}\middle| z\right]$$
$$- z(3-2z)\frac{(c+1)}{(c+3)}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+2\\ c+1, c+4\end{matrix}\middle| z\right]$$
$$+ 2z^2\frac{(c+1)(c+2)}{(c+3)(c+4)}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+3\\ c+1, c+5\end{matrix}\middle| z\right]$$
$$+ 2z^2(3-2z)\frac{c(c+2)}{(c+3)(c+4)}\,{}_4F_3\left[\begin{matrix}a, 1-a, c+1, c+3\\ c, c+2, c+5\end{matrix}\middle| z\right]$$
$$- 4z^3\frac{c(c+2)}{(c+4)(c+5)}\,{}_4F_3\left[\begin{matrix}a, 1-a, c+1, c+4\\ c, c+2, c+6\end{matrix}\middle| z\right].$$

If $a = 1$ in (5.47), we obtain

$$(1-z)^{c+2}{}_2F_1\left[\begin{matrix}c/2, c/2-1/2\\ c+3\end{matrix}\middle| 4z(1-z)\right] \tag{5.48}$$
$$= 1 - 6\frac{(c+1)}{(c+3)}z + 12\frac{(c^2+3c+1)}{(c+3)(c+4)}z^2 - 8\frac{c(c+2)}{(c+3)(c+5)}z^3.$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.48), we obtain the following summation theorem

$${}_4F_3\left[\begin{matrix}c/2, c/2-1/2, -n, \beta = A+2+n\\ c+3, \beta/2+a/2+1, \beta/2+a/2+3/2\end{matrix}\middle| 1\right] \tag{5.49}$$
$$= \frac{(\beta)_n}{(\beta+c+2)_n}\Big\{1 + 6\frac{(c+1)n}{(c+3)\beta} + 12\frac{(c^2+3c+1)n(n-1)}{(c+3)(c+4)}\beta(\beta+1)$$
$$- 8\frac{c(c+2)n(n-1)(n-2)}{(c+3)(c+5)\beta(\beta+1)(\beta+2)}\Big\}.$$

Case 5.15. *((5.42); (5.2); $g(z) = z(1-z)$, $h(z) = z$)*

$$(1-z)^{c+3}{}_2F_1\left[\begin{matrix}c/2-a/2, c/2+a/2-1/2\\ c+4\end{matrix}\middle| 4z(1-z)\right] \tag{5.50}$$
$$= (1-z){}_2F_1\left[\begin{matrix}a, 1-a\\ c+4\end{matrix}\middle| z\right] - 2z(1-z)\frac{c}{c+4}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+1\\ c, c+5\end{matrix}\middle| z\right]$$
$$- 2z(1-z)\frac{c+1}{c+4}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+2\\ c+1, c+5\end{matrix}\middle| z\right] - z(3-2z)\frac{c+2}{c+4}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+3\\ c+2, c+5\end{matrix}\middle| z\right]$$
$$+ 2z^2\frac{(c+2)(c+3)}{(c+4)(c+5)}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+4\\ c+2, c+6\end{matrix}\middle| z\right]$$
$$+ 4z^2(1-z)\frac{c(c+2)}{(c+4)(c+5)}\,{}_4F_3\left[\begin{matrix}a, 1-a, c+1, c+3\\ c, c+2, c+6\end{matrix}\middle| z\right]$$
$$+ 2z^2(3-2z)\frac{c(c+3)}{(c+4)(c+5)}\,{}_4F_3\left[\begin{matrix}a, 1-a, c+1, c+4\\ c, c+3, c+6\end{matrix}\middle| z\right]$$
$$+ 2z^2(3-2z)\frac{(c+1)(c+3)}{(c+4)(c+5)}\,{}_4F_3\left[\begin{matrix}a, 1-a, c+2, c+4\\ c+1, c+3, c+6\end{matrix}\middle| z\right]$$
$$- 4z^3\frac{c(c+3)}{(c+5)(c+6)}\,{}_4F_3\left[\begin{matrix}a, 1-a, c+1, c+5\\ c, c+3, c+7\end{matrix}\middle| z\right]$$
$$- 4z^3\frac{(c+1)(c+3)}{(c+5)(c+6)}\,{}_4F_3\left[\begin{matrix}a, 1-a, c+2, c+5\\ c+1, c+3, c+7\end{matrix}\middle| z\right]$$
$$- 4z^3(3-2z)\frac{c(c+2)}{(c+5)(c+6)}\,{}_5F_4\left[\begin{matrix}a, 1-a, c+1, c+3, c+5\\ c, c+2, c+4, c+7\end{matrix}\middle| z\right]$$
$$+ 8z^4\frac{c(c+2)}{(c+6)(c+7)}\,{}_5F_4\left[\begin{matrix}a, 1-a, c+1, c+3, c+6\\ c, c+2, c+4, c+8\end{matrix}\middle| z\right].$$

If $a = 1$ in (5.50), we obtain the special case

$$(1-z)^{c+3}{}_2F_1\left[\begin{matrix}c/2, c/2-1/2\\ c+4\end{matrix}\middle| 4z(1-z)\right] = 1 - 4\frac{(2c+3)}{(c+4)}z \tag{5.51}$$
$$+ 12\frac{(2c^2+8c+5)}{(c+4)(c+5)}z^2 - 8\frac{(2c+5)(2c^2+10c+3)}{(c+4)(c+5)(c+6)}z^3 + 16\frac{c(c+2)}{(c+5)(c+7)}z^4$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.51), we obtain the following summation theorem

$${}_4F_3\left[\begin{matrix}c/2, c/2-1/2, -n, \beta+c+3+n\\ c+4, \beta/2+c/2+3/2, \beta/2+c/2+2\end{matrix}\middle| 1\right] = \frac{(\beta)_n}{(\beta+c+3)_n}\Big\{1 + 4\frac{(2c+3)n}{(c+4)\beta} \tag{5.52}$$
$$+ 12\frac{(2c^2+8c+5)n(n-1)}{(c+4)(c+5)\beta(\beta+1)} + 8\frac{(2c+5)(2c^2+10c+3)n(n-1)(n-2)}{(c+4)(c+5)(c+6)\beta(\beta+1)(\beta+2)}$$
$$+ 16\frac{c(c+2)n(n-1)(n-2)(n-3)}{(c+5)(c+7)\beta(\beta+1)(\beta+2)(\beta+3)}\Big\}.$$

Case 5.16. *([19, Th. 32, p. 90]; (5.2); $g(z) = z(1-z)$, $h(z) = z/(1-z)$)*

$$(1-z)^{a-2}{}_3F_2\left[\begin{matrix}a/2, a/2-3/2, 1+a-b-c\\ 1+a-b, 1+a-c\end{matrix}\middle| 4z(1-z)\right] \tag{5.53}$$
$$= (1-z){}_4F_3\left[\begin{matrix}a/2-3/2, a, b, c\\ a/2+1/2, 1+a-b, 1+a-c\end{matrix}\middle| \frac{-z}{1-z}\right]$$

$$- \frac{(a-3)}{(a+1)}\Big(\frac{z}{1-z}\Big)\, {}_4F_3\left[\begin{matrix} a/2-1/2, a, b, c \\ a+3/2, 1+a-b, 1+a-c \end{matrix}\middle| \frac{-z}{1-z}\right] + z\frac{(a-3)(a-1)}{(a+1)(a+3)}\Big(\frac{z}{1-z}\Big)^2 {}_4F_3\left[\begin{matrix} a/2+1/2, a, b, c \\ a+5/2, 1+a-b, 1+a-c \end{matrix}\middle| \frac{-z}{1-z}\right].$$

If $c = 0$ in (5.53), we obtain

$$(1-z)^{a-2}{}_2F_1\left[\begin{matrix} a/2, a/2-3/2 \\ 1+a \end{matrix}\middle| 4z(1-z)\right] = 1 - z - \frac{(a-3)z}{(a+1)(1-z)} + \frac{(a-3)(a-1)z^3}{(a+1)(a+3)(1-z)^2}. \tag{5.54}$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.54), we obtain

$${}_4F_3\left[\begin{matrix} a/2, a/2-3/2, -n, \beta+a-2+n \\ 1+a, \beta/2+a/2-1, \beta/2+a/2-1/2 \end{matrix}\middle| 1\right] = \frac{(\beta)_n}{(\beta+a-2)_n} \Big\{1 + \frac{n}{\beta} + \frac{(a-3)n}{(a+1)(\beta+n-1)} - \frac{(a-3)(a-1)n(n-1)(n-2)}{(a+1)(a+3)\beta(\beta+n-2)(\beta+n-1)}\Big\}. \tag{5.55}$$

Case 5.17. *([6, Eq. (37), p. 119]; (5.2); $g(z) = z^2$, $h(z) = z/(1-z)$)*

$$(1-z)^{2a-2}{}_2F_1\left[\begin{matrix} a-3/2, a \\ b \end{matrix}\middle| z^2\right] = \frac{(2a-1)}{4(a-1)}(1-z){}_3F_2\left[\begin{matrix} 2a-3, 2a, b-1/2 \\ 2a-1, 2b-1 \end{matrix}\middle| \frac{-2z}{1-z}\right] + \frac{(2a-3)}{4(a-1)}(1+z){}_2F_1\left[\begin{matrix} 2a-2, b-1/2 \\ 2b-1 \end{matrix}\middle| \frac{-2z}{1-z}\right]. \tag{5.56}$$

If $z = -1$ in (5.56), we find a special case of the Watson summation theorem [23, Eq.(III-23) , p. 245]. With (3.17), we easily obtain the special case

$$(1-z)\ {}_2F_1\left[\begin{matrix} 2a-3, 2a \\ 2a-1 \end{matrix}\middle| \frac{-2z}{1-z}\right] = \Big\{\frac{1+z}{1-z}\Big\}^{2-2a}\Big\{1 - \frac{(2a-5)}{(2a-1)}z\Big\}. \tag{5.57}$$

If $b \to 1/2$ in (5.56) and using (5.57), we have

$$4(a-1)\ {}_2F_1\left[\begin{matrix} a-3/2, a \\ 1/2 \end{matrix}\middle| z^2\right] = (1+z)^{-2a}(2a-2+(4a-3)z+2az^2+z^3) + (1-z)^{-2a}(2a-2-(4a-3)z+2az^2-z^3). \tag{5.58}$$

If $z = 1/2$ in (5.58), we obtain

$${}_2F_1\left[\begin{matrix} a-3/2, a \\ 1/2 \end{matrix}\middle| \frac{1}{4}\right] = \frac{2^{2a-5}}{(a-1)}\Big\{(1+3^{2-2a})(4a-3) - 2\Big\}. \tag{5.59}$$

Case 5.18. *((5.56); (5.1); $g(z) = z^2$, $h(z) = z/(1-z)$)*

$$(1-z)^{2a-3}{}_3F_2\left[\begin{matrix} a-2, a-3/2, a \\ a-1, b \end{matrix}\middle| z^2\right] = \frac{(2a-1)}{4(a-1)}(1-z){}_3F_2\left[\begin{matrix} 2a-4, 2a, b-1/2 \\ 2a-1, 2b-1 \end{matrix}\middle| \frac{-2z}{1-z}\right] + \frac{(2a-3)}{4(a-1)}(1-z){}_3F_2\left[\begin{matrix} 2a-4, 2a-2, b-1/2 \\ 2a-3, 2b-1 \end{matrix}\middle| \frac{-2z}{1-z}\right] \tag{5.60}$$

$$+\frac{(a-2)}{(a-1)} z\, {}_2F_1\left[\begin{matrix} 2a-3, b-1/2 \\ 2b-1 \end{matrix}\middle| \frac{-2z}{1-z}\right].$$

Putting $z=-1$ in (5.60), manipulating the series on the right hand side and using the Gauss theorem and the contiguous relation (3.17), we obtain

$${}_3F_2\left[\begin{matrix} a-2, a-3/2, a \\ a-1, b \end{matrix}\middle| 1\right] = \frac{2^{1+2b-2a}(1+2a-2b+2ab-2a^2)\Gamma(b-2a+5/2)\Gamma(b)}{(a-1)\sqrt{\pi}\,\Gamma(2b-2a+3)}. \tag{5.61}$$

If $b \to 1/2$ in (5.60), with the help with (5.57),we have

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a-2, a-3/2, a \\ a-1, 1/2 \end{matrix}\middle| z^2\right] \\ &= \frac{1}{2}(1-\frac{z}{a-1})(1-z)^{3-2a} + \frac{1}{2}(1+\frac{z}{a-1})(1+z)^{3-2a}. \end{aligned} \tag{5.62}$$

If $z=\pm 1/2$ in (5.62), we obtain

$${}_3F_2\left[\begin{matrix} a-3/2, a, a-2 \\ 1/2, a-1 \end{matrix}\middle| \frac{1}{4}\right] = \frac{2^{2a-5}}{(a-1)}\left\{3^{3-2a}(2a-1)+(2a-3)\right\}. \tag{5.63}$$

Case 5.19. *([19, Th. 27, p. 69 ; Th. 20, p. 60];(5.1); $c \to c-a$, $g(z) = z/(1-z)^2$, $h(z) = z$)*

$$\begin{aligned} &(1-z)^{2-c}{}_3F_2\left[\begin{matrix} c/2-1, c/2-1/2, c/2-a \\ c/2, c-a \end{matrix}\middle| \frac{-4z}{(1-z)^2}\right] \\ &= {}_3F_2\left[\begin{matrix} c/2-1, c-1, a \\ c/2, c-a \end{matrix}\middle| z\right] + \frac{c-2}{c} z\, {}_3F_2\left[\begin{matrix} c/2, c-1, a \\ c/2+1, c-a \end{matrix}\middle| z\right]. \end{aligned} \tag{5.64}$$

If $a=0$ in (5.64), we have

$$(1-z)^{2-c}{}_2F_1\left[\begin{matrix} c/2-1, c/2-1/2 \\ c \end{matrix}\middle| \frac{-4z}{(1-z)^2}\right] = 1 + \frac{(c-2)}{c} z. \tag{5.65}$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.65), we obtain

$${}_4F_3\left[\begin{matrix} c/2-1, c/2-1/2, c/2, -n, c-\beta-1 \\ c, c/2-\beta/2-1/2-n/2, c/2-\beta/2-n/2 \end{matrix}\middle| 1\right] = \frac{(\beta)_n}{(\beta-c+2)_n}\left\{1-\frac{(c-2)n}{c\beta}\right\}. \tag{5.66}$$

Case 5.20. *([19, Th. 20, p. 60, Th. 27, p. 69]; (5.2); $c \to c-a$, $g(z) = z/(1-z)^2$, $h(z) = z$)*

$$\begin{aligned} &(1-z)^{4-c}{}_3F_2\left[\begin{matrix} c/2-2, c/2-a, c/2-1/2 \\ c/2, c-a \end{matrix}\middle| -\frac{4z}{(1-z)^2}\right] \\ &= {}_3F_2\left[\begin{matrix} c/2-2, a, c-1 \\ c/2, c-a \end{matrix}\middle| z\right] + \frac{c-4}{c} z(1-z)\, {}_3F_2\left[\begin{matrix} c/2-1, a, c-1 \\ c/2+1, c-a \end{matrix}\middle| z\right] \\ &\quad - z^3 \frac{(c-4)(c-2)}{c(c+2)}\, {}_3F_2\left[\begin{matrix} c/2, a, c-1 \\ c/2+2, c-a \end{matrix}\middle| z\right]. \end{aligned} \tag{5.67}$$

If $z=-1$ and $a=1$ in (5.67), we obtain

$${}_3F_2\left[\begin{matrix} c/2-2, c/2-1, c/2-1/2 \\ c/2, c-1 \end{matrix}\middle| 1\right] \tag{5.68}$$

$$= \frac{2^{c-4}}{c}\Big\{c(c-2)(c-4)[\psi(c/4+1/2)-\psi(c/4)] - 2c^2 + 10c - 4\Big\}.$$

If $a = 0$ in (5.67), we have

$$(1-z)^{4-c}{}_2F_1\left[\begin{matrix}c/2-2, c/2-1/2\\ c\end{matrix}\middle| -\frac{4z}{(1-z)^2}\right] \tag{5.69}$$
$$= 1 + \frac{(c-4)}{c}z - \frac{(c-4)}{c}z^2 - \frac{(c-4)(c-2)}{c(c+2)}z^3.$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ on both sides of (5.69), we obtain

$${}_4F_3\left[\begin{matrix}c/2-2, c/2-1/2, c/2, -n, c-\beta-3\\ c, c/2-\beta/2-3/2-n/2, c/2-\beta/2-1-n/2\end{matrix}\middle| 1\right] \tag{5.70}$$
$$= \frac{(\beta)_n}{(\beta-c+4)_n}\Big\{1 - \frac{(c-4)n}{c\beta} - \frac{(c-4)n(n-1)}{c\beta(\beta+1)} + \frac{(c-4)(c-2)n(n-1)(n-2)}{c(c+2)\beta(\beta+1)(\beta+2)}\Big\}.$$

Case 5.21. *([6, Eq. (45), p. 120]; (2.5); $\beta = \alpha + 4$, $h(z) = z/(2-z)$, $h(z) = z$)*

$$2^{a-4}(2-z)^{7-a}{}_2F_1\left[\begin{matrix}a/2-7/2, a/2\\ b+1/2\end{matrix}\middle| \Big(\frac{z}{2-z}\Big)^2\right] \tag{5.71}$$
$$= \frac{(a-1)(a-3)}{(a-4)(a-6)}{}_3F_2\left[\begin{matrix}a-7, a, b\\ a-3, 2b\end{matrix}\middle| z\right] + 3(1-z)\frac{(a-1)(a-7)}{(a-4)(a-6)}{}_3F_2\left[\begin{matrix}a-6, a, b\\ a-2, 2b\end{matrix}\middle| z\right]$$
$$+ 3(1-z)^2\frac{(a-1)(a-7)}{(a-2)(a-4)}{}_3F_2\left[\begin{matrix}a-5, a, b\\ a-1, 2b\end{matrix}\middle| z\right]$$
$$+ (1-z)^3\frac{(a-5)(a-7)}{(a-2)(a-4)}{}_2F_1\left[\begin{matrix}a-4, b\\ 2b\end{matrix}\middle| z\right].$$

If $z = 1$ in (5.71), we obtain

$${}_3F_2\left[\begin{matrix}a-7, a, b\\ a-3, 2b\end{matrix}\middle| 1\right] = \frac{2^{a-4}(a+4)(a-6)\Gamma(b+1/2)\Gamma(b+4-a)}{(a-1)(a-3)\Gamma(b-a/2+4)\Gamma(b-a/2+1/2)}, \tag{5.72}$$

a special case of the Watson summation theorem [23, Eq. (III-23), p. 245].

Case 5.22. *([6, Eq. (44), p. 120]; (5.1); $g(z) = z^2/(1-z)$, $h(z) = z$)*

$$(1-z)^{1-a/2}{}_2F_1\left[\begin{matrix}a/2-1, b-a/2\\ b+1/2\end{matrix}\middle| \frac{-z^2}{4(1-z)}\right] \tag{5.73}$$
$$= {}_3F_2\left[\begin{matrix}a, a-2, b\\ a-1, 2b\end{matrix}\middle| z\right] - \frac{(a-2)}{2(a-1)}z\,{}_2F_1\left[\begin{matrix}a-1, b\\ 2b\end{matrix}\middle| z\right]$$

If $a = -2n$ in (5.73), we find

$$\lim_{\epsilon\to 0}{}_3F_2\left[\begin{matrix}-2n+\epsilon, -2n-2, b\\ -2n-1+\epsilon, 2b\end{matrix}\middle| 2\right] = -\frac{(1/2)_n}{(2b+1)(b+3/2)_n}. \tag{5.74}$$

If $b \to 0$ in (5.73), we obtain after simplifications

$${}_2F_1\left[\begin{matrix}a/2-1, -a/2\\ 1/2\end{matrix}\middle| \frac{-z^2}{4(1-z)}\right] \tag{5.75}$$

$$= \frac{(2a-2-az)}{4(a-1)}(1-z)^{-a/2} + \frac{(2a-2-(3a-4)z+(a-2)z^2)}{4(a-1)}(1-z)^{a/2-2}.$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ *on the sides of (5.75), we obtain after simplifications the following summation theorem*

$$\begin{aligned}&{}_4F_3\left[\left.\begin{matrix}a/2-1,-a/2,-n/2,-n/2+1/2\\ 1/2,\beta,1-\beta-n\end{matrix}\right|1\right] \\ &= \frac{1}{4(a-1)(\beta)_n}\Big[(2a\beta-2\beta-a^2+a+an)(\beta-a/2+1)_{n-1} \\ &+(\beta+a/2-2+n)(2a\beta-2\beta+a^2-3a+an-2n+2)(\beta+a/2)_{n-2}\Big].\end{aligned} \tag{5.76}$$

Case 5.23. *([6, Eq. (44), p. 120]; (5.2); $g(z)=z^2/(1-z)$, $h(z)=z$)*

$$\begin{aligned}&(1-z)^{2-a/2}{}_2F_1\left[\left.\begin{matrix}a/2-2,b-a/2\\ b+1/2\end{matrix}\right|\frac{-z^2}{4(1-z)}\right] = \frac{(a-2)}{2(a-3)}{}_3F_2\left[\left.\begin{matrix}a,a-4,b\\ a-2,2b\end{matrix}\right|z\right] \\ &+\frac{(a-4)}{4(a-3)}(2-3z)\,{}_3F_2\left[\left.\begin{matrix}a-3,a,b\\ a-1,2b\end{matrix}\right|z\right] - \frac{(a-4)}{4(a-1)}z(1-z){}_2F_1\left[\left.\begin{matrix}a-2,b\\ 2b\end{matrix}\right|z\right].\end{aligned} \tag{5.77}$$

If $b\to 0$ in (5.77), we obtain

$$\begin{aligned}&(1-z)^{2-a/2}{}_2F_1\left[\left.\begin{matrix}a/2-2,-a/2\\ 1/2\end{matrix}\right|\frac{-z^2}{4(1-z)}\right] \\ &= \frac{1}{8(a-3)(a-1)}\Big\{4(a-1)(a-3)-2(a-4)(2a-3)z+(a-4)(a-3)z^2 \\ &+(1-z)^{2-a}\Big[12+4a(a-4)-2a(2a-5)z+a(a-1)z^2\Big]\Big\}.\end{aligned} \tag{5.78}$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ *on the sides of (5.78), we obtain after simplifications the following summation theorem*

$$\begin{aligned}&{}_4F_3\left[\left.\begin{matrix}a/2-2,-a/2,-n/2,-n/2+1/2\\ 1/2,-n-\beta-1+a/2,\beta+2-a/2\end{matrix}\right|1\right] \\ &= \frac{1}{8(a-1)(a-3)}\Bigg\{\frac{(\beta)_n}{(\beta+2-a/2)_n}\Big[4(a-3)(a-1) \\ &+2(a-4)(2a-3)\frac{n}{\beta}+(a-4)(a-3)\frac{n(n-1)}{\beta(\beta+1)}\Big] \\ &+\frac{(\beta-a+2)_n}{(\beta+2-a/2)_n}\Big[12+4a(a-4)+2a(2a-5)\frac{n}{(\beta+2-a)} \\ &+a(a-1)\frac{n(n-1)}{(\beta+2-a)(\beta+3-a)}\Big]\Bigg\}.\end{aligned} \tag{5.79}$$

Case 5.24. *([6, Eq. (44), p. 120]; (5.1); $g(z)=z^2/(1-z)$, $h(z)=z^2$)*

$$\begin{aligned}&(1-z)^{1-a/2}{}_2F_1\left[\left.\begin{matrix}a/2-1,b-a/2\\ b+1/2\end{matrix}\right|\frac{-z^2}{4(1-z)}\right] \\ &= {}_4F_3\left[\left.\begin{matrix}a/2-1,a/2+1/2,b/2,b/2+1/2\\ b,b+1/2,1/2\end{matrix}\right|z^2\right] \\ &+\frac{a(a-2)}{2(a-1)}z\,{}_4F_3\left[\left.\begin{matrix}a/2-1/2,a/2+1,b/2+1/2,b/2+1\\ b+1/2,b+1,3/2\end{matrix}\right|z^2\right]\end{aligned} \tag{5.80}$$

$$- \frac{a-2}{2(a-1)} z\, {}_4F_3\left[\begin{matrix} a/2, a/2-1/2, b/2, b/2+1/2 \\ b, b+1/2, 1/2 \end{matrix}\middle| z^2\right]$$
$$- \frac{a-2}{4} z^2\, {}_4F_3\left[\begin{matrix} a/2+1/2, a/2, b/2+1/2, b/2+1 \\ b+1/2, b+1, 3/2 \end{matrix}\middle| z^2\right].$$

Applying the operator ${}_zO_\beta^{-n}\{\}|_{z=1}$ *on both sides of (5.80) with* $a = -1$ *and* $b = 1$ *, we obtain after simplifications*

$$ {}_3F_2\left[\begin{matrix} -3/2, -n/2, -n/2+1/2 \\ \beta+3/2, -\beta-1/2-n \end{matrix}\middle| 1\right] \tag{5.81}$$
$$= \frac{(\beta)_n}{(\beta+3/2)_n}\Big\{1 + \frac{3n}{2\beta} + \frac{3n(n-1)}{4\beta(\beta+1)} + \frac{n(n-1)(n-2)}{8\beta(\beta+1)(\beta+2)}\Big\}.$$

6. Conclusion

In this paper, we give a list of twenty-four presumed new transformations involving the Gauss hypergeometric function with quadratic rational arguments. They are obtained from known transformation formulas of the hypergeometric function, for the most part in Goursat's thesis [6]. They are added to those of the article [26]. These transformation formulas were discovered by applying systematic method developed in [26] and involving the well-poised operator ${}_{g(z)}O_\beta^\alpha$ (introduced by the author several years ago [24]), which was defined with the fractional derivative and its representation using the Pochhammer contour integral. The formulas obtained and their applications clearly demonstrate the utility and efficiency of this operator. It is certainly a powerful tool to systematically obtain new formulas involving generalized hypergeometric functions as well as the special functions of mathematical physics. In future work, we will give a list of transformations of the hypergeometric functions which have cubic and higher degree rational arguments. We will search for other applications of this fractional operator, especially some new transformation formulas and summation theorems for hypergeometric functions, by using the transformation formula (2.3), among others.

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