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A Unified Family of Apostol-Bernoulli Based Poly-Daehee Polynomials

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Dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday

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Abstract

We introduce a unified family of Apostol-Bernoulli based poly-Daehee polynomials. Then we provide a number of formulas involving these unified polynomials such as differential formulas, addition formulas, summation formulas, and an implicit summation formula. The identities presented here, being general, are pointed out to yield the corresponding formulas associated with relatively simple polynomials. Further we provide several other polynomials similar to these unified polynomials.

Keywords:
Apostol-Bernoulli based poly-Daehee polynomials and numbers, Apostol-Euler based poly-Daehee polynomials and numbers, Apostol-Genocchi based poly-Daehee polynomials and numbers, Differential formula, Addition formula, Summation formula, Implicit summation formula

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1. Introduction and preliminaries

A number of polynomials have been extensively investigated and found many applications in a variety of fields such as mathematical physics, mathematics, statistics, probability, combinatorics and engineering (see, e.g., [1, 2, 4],[6]-[13]).

The polylogarithm function $\mathrm{Li}_n(z)$ is defined by

$$\mathrm{Li}_n(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^n} \quad (n \in \mathbb{N}/\{1\};\, z \in \mathbb{C},\, |z| \leq 1)\,. \tag{1.1}$$

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(see [19], [27, Section 2.4]; see also [6, 8]). Here and throughout, let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ be the sets of positive integers, integers and complex numbers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In particular, we define $\mathrm{Li}_1(z)$ by

$$\mathrm{Li}_1(z) = \sum_{m=1}^{\infty} \frac{z^m}{m} = -\log(1-z) \quad (|z|<1). \tag{1.2}$$

The Daehee polynomials of the first kind $\mathcal{D}_n(u)$ are defined by the following generating function (see [14])

$$(1+t)^u \frac{\log(1+t)}{t} := \sum_{n=0}^{\infty} \mathcal{D}_n(u) \frac{t^n}{n!}. \tag{1.3}$$

The Daehee numbers $\mathcal{D}_n := \mathcal{D}_n(0)$ are generated by

$$\frac{\log(1+t)}{t} := \sum_{n=0}^{\infty} \mathcal{D}_n \frac{t^n}{n!}. \tag{1.4}$$

The Daehee polynomials of the second kind $\widehat{\mathcal{D}}_n(u)$ are defined by means of the generating function (see [14, Eq. 2.21])

$$(1+t)^{1-u} \frac{\log(1+t)}{t} := \sum_{n=0}^{\infty} \widehat{\mathcal{D}}_n(u) \frac{t^n}{n!}. \tag{1.5}$$

Obviously $\widehat{\mathcal{D}}_n(u) = \mathcal{D}_n(1-u)$ $(n \in \mathbb{N}_0)$.

The poly-Daehee polynomials $\mathcal{D}_n^{(k)}(u)$ are defined by (see [20])

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u := \sum_{n=0}^{\infty} \mathcal{D}_n^{(k)}(u) \frac{t^n}{n!} \quad (k \in \mathbb{N}). \tag{1.6}$$

From (1.2) and (1.3), we find

$$\mathcal{D}_n^{(1)}(u) = \mathcal{D}_n(u) \quad (n \in \mathbb{N}_0). \tag{1.7}$$

The poly-Daehee numbers $\mathcal{D}_n^{(k)} := \mathcal{D}_n^{(k)}(0)$ are generated by

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})} = \sum_{n=0}^{\infty} \mathcal{D}_n^{(k)} \frac{t^n}{n!} \quad (k \in \mathbb{N}). \tag{1.8}$$

Bernoulli polynomials $B_n(u)$ are defined by

$$\frac{t}{\mathrm{e}^t - 1} \mathrm{e}^{ut} = \sum_{n=0}^{\infty} B_n(u) \frac{t^n}{n!} \quad (|t| < 2\pi) \tag{1.9}$$

(see, e.g., [27, Section 1.7]). Bernoulli numbers $B_n := B_n(0)$ are given by

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi). \tag{1.10}$$

The first few of the Bernoulli numbers are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \dots$$

$$B_{2n+1} = 0 \quad (n \in \mathbb{N}).$$

For further Daehee polynomials and Bernoulli polynomials, one may be referred (for example) to [4, 11, 12, 14, 15, 20, 24].

Bernoulli polynomials of the second kind $b_n(u)$ are defined by the generating function

$$\frac{t}{\log(1+t)}\,(1+t)^u := \sum_{n=0}^{\infty} b_n(u)\,\frac{t^n}{n!}. \tag{1.11}$$

Bernoulli numbers of the second kind $b_n := b_n(0)$ are given by

$$\sum_{n=0}^{\infty} b_n\,\frac{t^n}{n!} = \frac{t}{\log(1+t)}. \tag{1.12}$$

The first few of the Bernoulli numbers of the second kind are

$$b_0 = 1, \quad b_1 = \frac{1}{2}, \quad b_2 = -\frac{1}{12}, \quad b_3 = \frac{1}{24},$$

$$b_4 = \; -\frac{19}{720}, \quad b_4 = \; -\frac{19}{720}, \quad b_5 = \frac{3}{160}, \ldots$$

Poly-Bernoulli polynomials $\mathcal{B}_n^{(k)}(u)$ are defined by (see [6])

$$\frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{\mathrm{e}^t-1}\,\mathrm{e}^{ut} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}(u)\frac{t^n}{n!}. \tag{1.13}$$

Poly-Bernoulli numbers $\mathcal{B}_n^{(k)} := \mathcal{B}_n^{(k)}(0)$ are given by

$$\frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{\mathrm{e}^t-1} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}\frac{t^n}{n!}. \tag{1.14}$$

From (1.2), (1.9) and (1.13), we have

$$\mathcal{B}_n^{(1)}(u) = B_n(u) \quad \text{and} \quad \mathcal{B}_n^{(1)} = B_n \quad (n \in \mathbb{N}_0). \tag{1.15}$$

The generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(u;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined by means of the following generating function (see [23])

$$\left(\frac{t}{\lambda\mathrm{e}^t-1}\right)^{\alpha}\mathrm{e}^{ut} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(u;\lambda)\frac{t^n}{n!} \tag{1.16}$$

$$(|t+\ln\lambda| < 2\pi, \ |\ln\lambda| < 2\pi; \ 1^{\alpha} := 1).$$

Then $B_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0;\lambda)$ are called generalized Apostol-Bernoulli numbers of order $\alpha \in \mathbb{C}$. The generalized Bernoulli polynomials $B_n^{(\alpha)}(u) := \mathcal{B}_n^{(\alpha)}(u;1)$ of order α are given by

$$\left(\frac{t}{\mathrm{e}^t-1}\right)^{\alpha}\mathrm{e}^{ut} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(u)\frac{t^n}{n!} \quad (|t| < 2\pi; 1^{\alpha} := 1) \tag{1.17}$$

and

$$B_n^{(\alpha)} := B_n^{(\alpha)}(0) = \mathcal{B}_n^{(\alpha)}(0;1) \tag{1.18}$$

are called generalized Bernoulli numbers of order α.

Remark 1.1. Here we should discuss why the method of writing the region of convergence for the series in (1.16) including (1.19), (1.22), (1.25), (1.26), (1.28), (5.3), (5.4), (5.5), and (5.6) does not follow the conventional way. Since all of the above-referred series are centered at the origin $t = 0$, the region of convergence for each series is conventionally denoted by $|t| < R$, where the positive real number R depends on each series. But it may be difficult for the region of convergence involved in each of the above-mentioned series to denote conventionally by $|t| < R$ with some positive real number R. We may give some arguments why the region of convergence for each series is and can be provided by, for example, $|t + \ln\lambda| < 2\pi$ in (1.16) instead of $|t| < R$: Since $\lambda e^t - 1 = 0 \Leftrightarrow e^{t+\ln\lambda} = 1 \Leftrightarrow t_k + \ln\lambda = i2k\pi$ $(i = \sqrt{-1},\ k \in \mathbb{Z})$, the generating function in (1.16) may have isolated singular points at $t_k = -\ln\lambda + i2k\pi$ $(k \in \mathbb{Z})$. Yet the isolated singular point $t_0 = -\ln\lambda$ can be cancelled by the numerator factor t^α, that is, the generating function in (1.16) can be analytic at $t_0 = -\ln\lambda$ and hence analytic in the region $|t + \ln\lambda| < 2\pi$. Further, since $||t| - |\ln\lambda|| \leq |t + \ln\lambda| < 2\pi$, we have $|\ln\lambda| - 2\pi < |t| < |\ln\lambda| + 2\pi$. If $|\ln\lambda| \geq 2\pi$, then $|t| > 0$. When $|\ln\lambda| < 2\pi$, one finds that the generating function in (1.16) is analytic in a non-empty open disk, $D = \{t \in \mathbb{C} : |t| < R\}$ for some positive real number R to be determined, centered at the origin $t = 0$, which is composed of the points t satisfying $|t + \ln\lambda| < 2\pi$. Here, the radius of convergence R may not be determined *definitely* like $R = 2\pi$ in (1.9).

Some similar arguments as above may be given why the region of convergence for each other series cannot be recorded in the conventional way such as $|t| < R$. □

The generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(u;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined by the means of the generating function (see [21, 22])

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha e^{ut} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(u;\lambda)\frac{t^n}{n!} \tag{1.19}$$

$$(|t + \ln\lambda| < \pi,\ |\ln\lambda| < \pi;\ 1^\alpha := 1).$$

Then

$$\mathcal{E}_n^{(\alpha)}(u;1) := E_n^{(\alpha)}(u) \quad (n \in \mathbb{N}_0) \tag{1.20}$$

are the generalized Euler polynomials of order α (see, e.g., [27, p. 88]) and

$$\mathcal{E}_n^{(\alpha)}(0;\lambda) := \mathcal{E}_n^{(\alpha)}(\lambda) \tag{1.21}$$

are called Apostol-Euler numbers of order α.

The generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(u;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating function (see [21, 22])

$$\left(\frac{2t}{\lambda e^t + 1}\right)^\alpha e^{ut} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(u;\lambda)\frac{t^n}{n!} \tag{1.22}$$

$$(|t + \ln\lambda| < \pi,\ |\ln\lambda| < \pi;\ 1^\alpha := 1).$$

Then

$$\mathcal{G}_n^{(\alpha)}(u;1) = G_n^{(\alpha)}(u) \quad (n \in \mathbb{N}_0) \tag{1.23}$$

are the generalized Genocchi polynomials of order α and

$$\mathcal{G}_n^{(\alpha)}(0;\lambda) = \mathcal{G}_n^{(\alpha)}(\lambda) \quad (n \in \mathbb{N}_0) \tag{1.24}$$

are called Apostol-Genocchi numbers of order α.

We introduce a unified class of Apostol-Bernoulli based poly-Daehee polynomials ${}_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(u,v;\lambda)$ as follows:

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^u\left(\frac{t}{\lambda e^t - 1}\right)^\alpha e^{vt} = \sum_{n=0}^{\infty} {}_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(u,v;\lambda)\frac{t^n}{n!} \tag{1.25}$$

$$(k \in \mathbb{N};\ u,\ v,\ \alpha \in \mathbb{C};\ |t + \ln\lambda| < 2\pi,\ |\ln\lambda| < 2\pi,\ |t| < 1;\ 1^\alpha := 1).$$

We call ${}_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(\lambda) := {}_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(0,0;\lambda)$ Apostol-Bernoulli based poly-Daehee numbers which are generated by

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}\left(\frac{t}{\lambda e^t - 1}\right)^\alpha = \sum_{n=0}^{\infty} {}_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(\lambda)\frac{t^n}{n!} \tag{1.26}$$

$$\left(k \in \mathbb{N};\ \alpha \in \mathbb{C};\ |t + \ln \lambda| < 2\pi,\ |\ln \lambda| < 2\pi,\ |t| < 1;\ 1^{\alpha} := 1\right).$$

Also we denote

$$_{\mathcal{B}}\mathcal{D}_{n,\alpha}(u, v; \lambda) := {_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(1)}}(u, v; \lambda) \quad (n \in \mathbb{N}_0)\,. \tag{1.27}$$

Then we aim to investigate certain interesting properties of the unified polynomials (1.25) such as differential formulas, addition formulas, summation formulas, and an implicit summation formula. The identities presented here, being general, are indicated to yield the corresponding formulas associated with relatively simple polynomials. Further we provide several other polynomials similar to these unified polynomials.

Some obvious special cases of the unified polynomials (1.25) are considered in the following remark.

Remark 1.2.

(i) We call the case $k = 1$ of (1.25) Apostol-Bernoulli based Daehee polynomials $_{\mathcal{B}}\mathcal{D}_{n,\alpha}(u, v; \lambda)$:

$$\frac{\log(1+t)}{t}(1+t)^u \left(\frac{t}{\lambda \mathrm{e}^t - 1}\right)^{\alpha} \mathrm{e}^{vt} = \sum_{n=0}^{\infty} {_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}}(u, v; \lambda) \frac{t^n}{n!} \tag{1.28}$$

$$\left(u,\ v,\ \alpha \in \mathbb{C};\ |t + \ln \lambda| < 2\pi,\ |\ln \lambda| < 2\pi,\ |t| < 1;\ 1^{\alpha} := 1\right).$$

(ii) $_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(u, v; \lambda) = \mathcal{D}_n^{(k)}(u)$ are poly-Daehee polynomials.

(iii) $_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(1)}(u, v; \lambda) = \mathcal{D}_n(u)$ are Daehee polynomials.

(iv) Setting $\alpha = 0 = v$, $k = 1$, $t = \xi t$ in (1.25), we have the n-th twisted Daehee polynomials $\mathcal{D}_{n,\xi}(u)$ (see [24])

$$\frac{\log(1+\xi t)}{\xi t}\,(1+\xi t)^u = \sum_{n=0}^{\infty} \mathcal{D}_{n,\xi}(u) \frac{t^n}{n!}.$$

(v) $_{\mathcal{B}}\mathcal{D}_{n,0}^{(1)}(1-u, 0; \lambda) = \widehat{\mathcal{D}}_n(u)$ are the Daehee polynomials of the second kind. □

2. Differential formulas

We give partial differential formulas with respect to the parameters u, v and λ, respectively, as in the following theorem.

Theorem 2.1. *Each of the following formulas holds.*

$$\frac{\partial}{\partial u} {_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}}(u, v; \lambda) = \sum_{m=0}^{n-1} \frac{(-1)^{n+1-m}\, n!}{(n-m)\, m!}\, {_{\mathcal{B}}\mathcal{D}_{m,\alpha}^{(k)}}(u, v; \lambda) \tag{2.1}$$

$$(n \in \mathbb{N}_0,\ k \in \mathbb{N})\,;$$

$$\frac{\partial^m}{\partial v^m} {_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}}(u, v; \lambda) = \frac{n!}{(n-m)!}\, {_{\mathcal{B}}\mathcal{D}_{n-m,\alpha}^{(k)}}(u, v; \lambda) \tag{2.2}$$

$$(m,\ n \in \mathbb{N}_0;\ k \in \mathbb{N})\,;$$

$$\frac{\partial}{\partial \lambda} {_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}}(u, v; \lambda) = -\frac{\alpha}{n+1}\, {_{\mathcal{B}}\mathcal{D}_{n+1,\alpha+1}^{(k)}}(u, v+1; \lambda) \tag{2.3}$$

$$(n \in \mathbb{N}_0;\ k \in \mathbb{N})\,;$$

$$\frac{\partial^2}{\partial v \partial \lambda} {_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}}(u, v; \lambda) = -\alpha\, {_{\mathcal{B}}\mathcal{D}_{n,\alpha+1}^{(k)}}(u, v+1; \lambda) \tag{2.4}$$

$$(n \in \mathbb{N}_0;\ k \in \mathbb{N})\,.$$

Here and throughout, an empty sum is assumed to be nil.

Proof. Differentiating both sides of (1.25) with respect to u, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\partial}{\partial u} {}_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(u,v;\lambda)\frac{t^n}{n!} \\
&= \log(1+t)\cdot \frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u \left(\frac{t}{\lambda \mathrm{e}^t-1}\right)^{\alpha} \mathrm{e}^{vt} \\
&= t\left\{\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\, t^n\right\}\left\{\sum_{m=0}^{\infty} {}_{\mathcal{B}}\mathcal{D}_{m,\alpha}^{(k)}(u,v;\lambda)\frac{t^m}{m!}\right\} \\
&= t\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} \frac{(-1)^{n-m}}{(n-m+1)\,m!}\, {}_{\mathcal{B}}\mathcal{D}_{m,\alpha}^{(k)}(u,v;\lambda)\right\} t^n.
\end{aligned} \tag{2.5}$$

Equating the coefficients of t^n both sides of the first and last summations in (2.5), we get (2.1).

Similarly we can prove (2.2), (2.3) and (2.4). We omit the details.

□

3. Addition formulas and some other summation formulas

We provide addition formulas with respect to the parameters u, v and α, respectively, and some other summation formulas, as in the following theorem.

Theorem 3.1. *Each of the following formulas holds.*

$$\begin{aligned}
{}_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(u_1+u_2,v;\lambda) &= \sum_{m=0}^{n} \frac{n!}{m!}\binom{u_1}{n-m} {}_{\mathcal{B}}\mathcal{D}_{m,\alpha}^{(k)}(u_2,v;\lambda) \\
&= \sum_{m=0}^{n} \frac{n!}{m!}\binom{u_2}{n-m} {}_{\mathcal{B}}\mathcal{D}_{m,\alpha}^{(k)}(u_1,v;\lambda) \\
&\qquad (n\in\mathbb{N}_0,\ k\in\mathbb{N});
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
{}_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(u,v_1+v_2;\lambda) &= \sum_{m=0}^{n} \binom{n}{m} v_1^{n-m}\, {}_{\mathcal{B}}\mathcal{D}_{m,\alpha}^{(k)}(u,v_2;\lambda) \\
&= \sum_{m=0}^{n} \binom{n}{m} v_2^{n-m}\, {}_{\mathcal{B}}\mathcal{D}_{m,\alpha}^{(k)}(u,v_1;\lambda) \\
&\qquad (n\in\mathbb{N}_0,\ k\in\mathbb{N});
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
{}_{\mathcal{B}}\mathcal{D}_{n,\alpha_1+\alpha_2}^{(k)}(u,v;\lambda) &= \sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{n-m}^{(\alpha_1)}(\lambda)\, {}_{\mathcal{B}}\mathcal{D}_{m,\alpha_2}^{(k)}(u,v;\lambda) \\
&= \sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{n-m}^{(\alpha_2)}(\lambda)\, {}_{\mathcal{B}}\mathcal{D}_{m,\alpha_1}^{(k)}(u,v;\lambda) \\
&\qquad (n\in\mathbb{N}_0,\ k\in\mathbb{N});
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u_1+u_2, v_1+v_2;\lambda)\\
&\quad=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{u_1}{n-m}\frac{n!\,v_1^{m-\ell}}{(m-\ell)!\,\ell!}\,{}_{\mathcal{B}}\mathcal{D}^{(k)}_{\ell,\alpha}(u_2,v_2;\lambda)\\
&\quad=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{u_1}{n-m}\frac{n!\,v_2^{m-\ell}}{(m-\ell)!\,\ell!}\,{}_{\mathcal{B}}\mathcal{D}^{(k)}_{\ell,\alpha}(u_2,v_1;\lambda)\\
&\quad=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{u_2}{n-m}\frac{n!\,v_1^{m-\ell}}{(m-\ell)!\,\ell!}\,{}_{\mathcal{B}}\mathcal{D}^{(k)}_{\ell,\alpha}(u_1,v_2;\lambda)\\
&\quad=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{u_2}{n-m}\frac{n!\,v_2^{m-\ell}}{(m-\ell)!\,\ell!}\,{}_{\mathcal{B}}\mathcal{D}^{(k)}_{\ell,\alpha}(u_1,v_1;\lambda)\\
&\qquad\qquad (n\in\mathbb{N}_0,\ k\in\mathbb{N});
\end{aligned}
\tag{3.4}
$$

$$
{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u,v;\lambda)=\sum_{n=0}^{\infty}\binom{n}{m}\mathcal{D}^{(k)}_{n-m}(u)\mathcal{B}^{(\alpha)}_{m}(v;\lambda)\qquad (n\in\mathbb{N}_0,\ k\in\mathbb{N});
\tag{3.5}
$$

$$
\sum_{m=0}^{n}\binom{n}{m}B_m\,{}_{\mathcal{B}}\mathcal{D}_{n-m,\alpha}(u,v;\lambda)=\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}^{(k)}_{m}\,{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n-m,\alpha}(u,v;\lambda)\qquad (n\in\mathbb{N}_0,\ k\in\mathbb{N});
\tag{3.6}
$$

$$
\mathcal{B}^{(\alpha)}_{n}(v;\lambda)=\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}^{(k)}_{m}(-u)\,{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n-m,\alpha}(u,v;\lambda)\qquad (n\in\mathbb{N}_0,\ k\in\mathbb{N});
\tag{3.7}
$$

$$
{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u,v;\lambda)=\sum_{m=0}^{n}\binom{n}{m}\mathcal{D}^{(k)}_{n-m,\alpha}(u,v-\nu)\,\mathcal{B}^{(\tau)}_{m}(\nu;\lambda)\qquad (n\in\mathbb{N}_0,\ k\in\mathbb{N});
\tag{3.8}
$$

$$
\begin{aligned}
&{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha_1+\alpha_2}(u,v_1+v_2;\lambda)\\
&\quad=\sum_{m=0}^{n}\binom{n}{m}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n-m,\alpha_1}(u,v_1+v_2;\lambda)\,\mathcal{B}^{(\alpha_2)}_{m}(v_2;\lambda)\\
&\qquad\qquad (n\in\mathbb{N}_0,\ k\in\mathbb{N}).
\end{aligned}
\tag{3.9}
$$

Proof. Replacing u in (1.25) by u_1+u_2, we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u_1+u_2,v;\lambda)\frac{t^n}{n!}\\
&\quad=(1+t)^{u_1}\cdot\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^{u_2}\left(\frac{t}{\lambda\mathrm{e}^t-1}\right)^{\alpha}\ \mathrm{e}^{vt}\\
&\quad=\left\{\sum_{n=0}^{\infty}\binom{u_1}{n}t^n\right\}\left\{\sum_{m=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{m,\alpha}(u_2,v;\lambda)\frac{t^m}{m!}\right\}\\
&\quad=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{u_1}{n-m}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{m,\alpha}(u_2,v;\lambda)\,\frac{t^n}{m!}.
\end{aligned}
\tag{3.10}
$$

Equating the coefficients of t^n on both sides of the first and last summations of (3.10), we obtain the first equality of (3.1). Changing the roles of u_1 and u_2 in the above proof, we find the second equality of (3.1).

By using (1.25) and (1.14), we find

$$\begin{aligned}&\frac{\log(1+t)}{\mathrm{e}^t-1}(1+t)^u\left(\frac{t}{\lambda\mathrm{e}^t-1}\right)^\alpha \mathrm{e}^{vt}\\&\quad=\left\{\frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{\mathrm{e}^t-1}\right\}\left\{\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u\left(\frac{t}{\lambda\mathrm{e}^t-1}\right)^\alpha \mathrm{e}^{vt}\right\}\\&\quad=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_m^{(k)}\,{}_{\mathcal{B}}\mathcal{D}_{n-m,\alpha}^{(k)}(u,v;\lambda)\right\}\frac{t^n}{n!}.\end{aligned} \tag{3.11}$$

On the other hand, using (1.15) and (1.27), we get

$$\begin{aligned}&\frac{\log(1+t)}{\mathrm{e}^t-1}(1+t)^u\left(\frac{t}{\lambda\mathrm{e}^t-1}\right)^\alpha \mathrm{e}^{vt}\\&\quad=\left\{\frac{t}{\mathrm{e}^t-1}\right\}\left\{\frac{\log(1+t)}{t}(1+t)^u\left(\frac{t}{\lambda\mathrm{e}^t-1}\right)^\alpha \mathrm{e}^{vt}\right\}\\&\quad=\left\{\sum_{m=0}^{\infty}B_m\frac{t^m}{m!}\right\}\left\{\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}_{n,\alpha}(u,v;\lambda)\frac{t^n}{n!}\right\}\\&\quad=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}B_m\,{}_{\mathcal{B}}\mathcal{D}_{n-m,\alpha}(u,v;\lambda)\right\}\frac{t^n}{n!}.\end{aligned} \tag{3.12}$$

Equating the coefficients of t^n on both sides of the last summations of (3.11) and (3.12), we obtain (3.6).

We use (1.16) and (1.25) to combine

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(v;\lambda)\frac{t^n}{n!}=\left(\frac{t}{\lambda\mathrm{e}^t-1}\right)^\alpha \mathrm{e}^{vt}\\&\quad=\frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{\log(1+t)}(1+t)^{-u}\cdot\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u\left(\frac{t}{\lambda\mathrm{e}^t-1}\right)^\alpha \mathrm{e}^{vt}\\&\quad=\left\{\sum_{m=0}^{\infty}\mathcal{B}_m^{(k)}(-u)\frac{t^m}{m!}\right\}\left\{\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}_{n,\alpha}^{(k)}(u,v;\lambda)\frac{t^n}{n!}\right\}\\&\quad=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_m^{(k)}(-u)\,{}_{\mathcal{B}}\mathcal{D}_{n-m,\alpha}^{(k)}(u,v;\lambda)\right\}\frac{t^n}{n!}.\end{aligned} \tag{3.13}$$

Equating the coefficients of t^n of the first and last summations of (3.13), we obtain (3.7).

Similarly we can prove the other formulas. We omit the details.

□

4. An implicit summation formula

We give an implicit summation formula for the unified polynomials (1.25). To do this, we recall the following series manipulation formula

$$\sum_{N=0}^{\infty}g(N)\frac{(u+v)^N}{N!}=\sum_{n,m=0}^{\infty}g(n+m)\frac{u^n}{n!}\frac{v^m}{m!}. \tag{4.1}$$

Here $g:\mathbb{N}_0\to\mathbb{C}$ (or $\mathbb{R}$) is a function and u, v are real or complex numbers.

Theorem 4.1. *Let $k \in \mathbb{N}$, n, $m \in \mathbb{N}_0$. Also let u, v, w, α, $\lambda \in \mathbb{C}$ with $|\ln \lambda| < 2\pi$. Then the following implicit formula holds.*

$$\begin{aligned} &{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n+m,\alpha}(u,v;\lambda) \\ &\quad = \sum_{a=0}^{n}\sum_{b=0}^{m}\binom{n}{a}\binom{m}{b}(v-w)^{a+b}\,{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n+m-a-b,\alpha}(u,w;\lambda). \end{aligned} \tag{4.2}$$

Proof. Rewrite (1.25) as follows:

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u\left(\frac{t}{\lambda \mathrm{e}^t-1}\right)^{\alpha} = \mathrm{e}^{-vt}\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u,v;\lambda)\frac{t^n}{n!}. \tag{4.3}$$

Since the left-member of (4.3) is independent of the parameter v, we can substitute any parameter, say w, for the parameter v in the right-member of (4.3). Then

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u\left(\frac{t}{\lambda \mathrm{e}^t-1}\right)^{\alpha} = \mathrm{e}^{-wt}\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u,w;\lambda)\frac{t^n}{n!}. \tag{4.4}$$

Equating the right-members of (4.3) and (4.4), we have

$$\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u,v;\lambda)\frac{t^n}{n!} = \mathrm{e}^{(v-w)t}\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u,w;\lambda)\frac{t^n}{n!}. \tag{4.5}$$

Replacing t by $t_1 + t_2$ in (4.5), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u,v;\lambda)\frac{(t_1+t_2)^n}{n!} \\ &\quad = \mathrm{e}^{(v-w)(t_1+t_2)}\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n,\alpha}(u,w;\lambda)\frac{(t_1+t_2)^n}{n!}. \end{aligned} \tag{4.6}$$

Using (4.1) in (4.6), we get

$$\begin{aligned} &\sum_{n,\,m=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n+m,\alpha}(u,v;\lambda)\,\frac{t_1^n}{n!}\,\frac{t_2^m}{m!} \\ &\quad = \mathrm{e}^{(v-w)(t_1+t_2)}\sum_{n,\,m=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n+m,\alpha}(u,w;\lambda)\,\frac{t_1^n}{n!}\,\frac{t_2^m}{m!}. \end{aligned} \tag{4.7}$$

Using (4.1), we find

$$\begin{aligned} \mathrm{e}^{(v-w)(t_1+t_2)} &= \sum_{n=0}^{\infty}(v-w)^n\,\frac{(t_1+t_2)^n}{n!} \\ &= \sum_{a,\,b=0}^{\infty}(v-w)^{a+b}\,\frac{t_1^a}{a!}\,\frac{t_2^b}{b!}. \end{aligned} \tag{4.8}$$

Setting the right-member of (4.8) in the right-hand side of (4.7) and coupling the quadruple series in two pairs with indices (n, a) and (m, b), and rearranging the double series in each pair, we derive

$$\begin{aligned} &\mathrm{e}^{(v-w)(t_1+t_2)}\sum_{n,\,m=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n+m,\alpha}(u,w;\lambda)\,\frac{t_1^n}{n!}\,\frac{t_2^m}{m!} \\ &= \sum_{n,\,m=0}^{\infty}{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n+m,\alpha}(u,w;\lambda)\,\frac{t_1^n}{n!}\,\frac{t_2^m}{m!}\sum_{a,\,b=0}^{\infty}(v-w)^{a+b}\,\frac{t_1^a}{a!}\,\frac{t_2^b}{b!} \\ &= \sum_{n,\,m=0}^{\infty}\sum_{a=0}^{n}\sum_{b=0}^{m}(v-w)^{a+b}\,{}_{\mathcal{B}}\mathcal{D}^{(k)}_{n+m-a-b,\alpha}(u,w;\lambda) \\ &\qquad\qquad \times \frac{t_1^n\,t_2^m}{(n-a)!a!\,(m-b)!b!}. \end{aligned} \tag{4.9}$$

Putting the last member of (4.9) in the right-hand side of (4.7) and equating the coefficients of $t_1^n\, t_2^m$ on both sides of the resulting identity, we obtain (4.2).

□

5. Remarks

The results given here, being general, enable us to give a variety of known and new identities involving relatively simple polynomials (see, e.g., Remark 1.2). For example, the particular case $\lambda = 1 = \alpha$ of the unified polynomials (1.25) can yield new identities, corresponding to those provided in this paper, involving Bernoulli based poly-Daehee polynomials ${}_{\mathcal{B}}\mathcal{D}_n^{(k)}(u, v) := {}_{\mathcal{B}}\mathcal{D}_{n,1}^{(k)}(u, v; 1)$ and Bernoulli based poly-Daehee numbers ${}_{\mathcal{B}}\mathcal{D}_n^{(k)}(u) := {}_{\mathcal{B}}\mathcal{D}_{n,1}^{(k)}(u, 0; 1)$, which are given, respectively, as follows:

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u \left(\frac{t}{\mathrm{e}^t - 1}\right) \mathrm{e}^{vt} = \sum_{n=0}^{\infty} {}_{\mathcal{B}}\mathcal{D}_n^{(k)}(u, v)\frac{t^n}{n!} \tag{5.1}$$

$$(k \in \mathbb{N};\ u,\ v \in \mathbb{C};\ |t| < 1;\ 1^u := 1)$$

and

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u\, \frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} {}_{\mathcal{B}}\mathcal{D}_n^{(k)}(u)\frac{t^n}{n!} \tag{5.2}$$

$$(k \in \mathbb{N};\ u \in \mathbb{C};\ |t| < 1;\ 1^u := 1).$$

Also we can consider several other polynomials similar to the unified polynomials (1.25):

(i) Apostol-Euler based poly-Daehee polynomials ${}_{\mathcal{E}}\mathcal{D}_{n,\alpha}^{(k)}(u, v; \lambda)$

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u \left(\frac{2}{\lambda \mathrm{e}^t + 1}\right)^{\alpha} \mathrm{e}^{vt} = \sum_{n=0}^{\infty} {}_{\mathcal{E}}\mathcal{D}_{n,\alpha}^{(k)}(u, v; \lambda)\frac{t^n}{n!} \tag{5.3}$$

$$(k \in \mathbb{N};\ u,\ v,\ \alpha \in \mathbb{C};\ |t + \ln \lambda| < \pi,\ |\ln \lambda| < \pi,\ |t| < 1;\ 1^{\alpha} := 1).$$

(ii) Apostol-Genocchi based poly-Daehee polynomials ${}_{\mathcal{G}}\mathcal{D}_{n,\alpha}^{(k)}(u, v; \lambda)$

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-\mathrm{e}^{-t})}(1+t)^u \left(\frac{2t}{\lambda \mathrm{e}^t - 1}\right)^{\alpha} \mathrm{e}^{vt} = \sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{n,\alpha}^{(k)}(u, v; \lambda)\frac{t^n}{n!} \tag{5.4}$$

$$(k \in \mathbb{N};\ u,\ v,\ \alpha \in \mathbb{C};\ |t + \ln \lambda| < 2\pi,\ |\ln \lambda| < 2\pi,\ |t| < 1;\ 1^{\alpha} := 1).$$

(iii) Apostol-Euler based Daehee polynomials ${}_{\mathcal{E}}\mathcal{D}_{n,\alpha}(u, v; \lambda)$

$$\frac{\log(1+t)}{t}(1+t)^u \left(\frac{2}{\lambda \mathrm{e}^t + 1}\right)^{\alpha} \mathrm{e}^{vt} = \sum_{n=0}^{\infty} {}_{\mathcal{E}}\mathcal{D}_{n,\alpha}^{(k)}(u, v; \lambda)\frac{t^n}{n!} \tag{5.5}$$

$$(u,\ v,\ \alpha \in \mathbb{C};\ |t + \ln \lambda| < \pi,\ |\ln \lambda| < \pi,\ |t| < 1;\ 1^{\alpha} := 1).$$

(iv) Apostol-Genocchi based Daehee polynomials ${}_{\mathcal{G}}\mathcal{D}_{n,\alpha}(u, v; \lambda)$

$$\frac{\log(1+t)}{t}(1+t)^u \left(\frac{2t}{\lambda \mathrm{e}^t - 1}\right)^{\alpha} \mathrm{e}^{vt} = \sum_{n=0}^{\infty} {}_{\mathcal{G}}\mathcal{D}_{n,\alpha}^{(k)}(u, v; \lambda)\frac{t^n}{n!} \tag{5.6}$$

$$(u,\ v,\ \alpha \in \mathbb{C};\ |t + \ln \lambda| < 2\pi,\ |\ln \lambda| < 2\pi,\ |t| < 1;\ 1^{\alpha} := 1).$$

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References

[1] S. Araci and M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. **22 (3)**, 399-406, 2012.
[2] L. Carlitz, *A note on Bernoulli and Euler polynomials of the second kind*, Scripta Math. **25**, 323-330, 1961.
[3] L. Comtet and L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions (Translated from the French by J. W. Nienhuys)*, Reidel, Dordrecht and Boston, 1974.
[4] G. Dattoli, S. Lorenzutta and C. Cesarano, *Finite sums and generalized forms of Bernoulli polynomials*, Rend. Math. (Ser. 8) **19**, 385-391, 1999.
[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions Vol.I*, McGraw-Hill Book Company, New York, Toronto and London, 1953.
[6] M. Kaneko, *Poly-Bernoulli numbers*, J. Théor. Nombres Bordeaux **9 (1)**, 221-228, 1997.
[7] N.U. Khan and T. Usman, *A new class of Laguerre poly-Bernoulli numbers and polynomials*, Adv. Stud. Contemp. Math. **27 (2)**, 229–241, 2017.
[8] N.U. Khan, T. Usman and M. Aman, *Generating functions for Legendre-based poly-Bernoulli numbers and polynomials*, Honam Math. J. **39 (2)**, 217-231, 2017.
[9] N.U. Khan, T. Usman and J. Choi, *A new class of generalized polynomials*, Turk. J. Math. **42**, 1366-1379, 2018.
[10] T. Kim, *On the multiple q-Genocchi and Euler numbers*, Russ. J. Math. Phys. **15 (4)**, 481-486, 2008.
[11] D.S. Kim, T. Kim, S.-H. Lee and J.-J. Seo, *Higher-order Daehee numbers and polynomials*, Int. J. Math. Anal. (Ruse) **8 (6)**, 273-283, 2014.
[12] T. Kim, S.-H. Lee, T. Mansour and J.-J. Seo, *A note on q-Daehee polynomials and numbers*, Adv. Stud. Contemp. Math. **24 (2)**, 155-160, 2014.
[13] T. Kim, H.I. Kwon, S.-H. Lee and J.-J. Seo, *A note on poly-Bernoulli numbers and polynomials of the second kind*, Adv. Differen. Equ. **2014**, Article Id 219, 2014.
[14] D.S. Kim and T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci. **7 (120)**, 5969-5976, 2013.
[15] D.S. Kim and T. Kim, *A study on the integral of the product of several Bernoulli polynomials*, Rocky Mountain J. Math. **44 (4)**, 1251-1263, 2014.
[16] D.S. Kim and T. Kim, *Some identities involving Genocchi polynomials and numbers*, Ars Comb. **121**, 403-412, 2015.
[17] D.S. Kim, N. Lee, J. Na and K.H. Park, *Identities of symmetry for higher order Euler polynomials in three variables (I)*, Adv. Stud. Contemp. Math. **22**, 51-74, 2012.
[18] T. Komatsu and F. Luca, *Some relationships between poly-Cauchy numbers and poly-Bernoulli numbers*, Ann. Math. Inform. **41**, 99-105, 2013.
[19] L. Lewin, *Polylogarithms and Associated Functions*, Elsevier (North-Holland), New York, London and Amsterdam, 1981.
[20] D.S. Lim and J. Kwon, *A note on poly-Daehee numbers and polynomials*, Proc. Jangjeon Math. Soc. **19 (2)**, 219-224, 2016.
[21] Q.-M. Luo, *Extension for the Genocchi polynomials and its Fourier expansions and integral representations*, Osaka J. Math. **48**, 291-309, 2011.
[22] Q.-M. Luo, *Apostol-Euler polynomials of higher order and the Gaussian hypergeometric function*, Taiwanese J. Math. **10 (4)**, 917-925, 2006.
[23] Q.-M. Luo and H. M. Srivastava, *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials*, J. Math. Anal. Appl. **308 (1)**, 290-302, 2005.
[24] J.-W. Park, S.-H. Rim and J. Kwon, *The twisted Daehee numbers and polynomials*, Adv. Differ. Equ. **9 (1)**, Article Id 1, 2014.
[25] E.D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.
[26] J.Sándor and B. Crstici, *Handbook of Number Theory II*, Kluwer Acad. Publ., Dordrecht, 2004.
[27] H.M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.