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A Family of Theta-Function Identities Based Upon q-Binomial Theorem and Heine's Transformations

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Abstract

The authors establish a set of two presumably new theta-function identities which are based upon q-binomial theorem and Heine's transformations. Several closely-related identities such as (for example) q-product identities and Jacobi's triple-product identity are also considered.

Keywords: Jacobi's triple-product identity; q-Product identities, Euler's Pentagonal Number Theorem, q-Binomial theorem, Heine's transformations

2010 MSC: 11A41, 11E25, 33D15, 33E20

1. Introduction

Throughout this article, we denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers, respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}$$

and recall the following q-notations (see, for example, [13, Chapter 3, Section 3.2.1], [19, Chapter 6] and [20, p. 346]).

The q-shifted factorial $(a; q)_n$ is defined (for $|q| < 1$) by

$$(a;q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1} (1 - aq^k) & (n \in \mathbb{N}), \end{cases} \tag{1}$$

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where $a, q \in \mathbb{C}$ and it is assumed *tacitly* that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$. We also write

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) = \prod_{k=1}^{\infty}(1-aq^{k-1}) \qquad (a,q \in \mathbb{C};\ |q|<1). \tag{2}$$

It should be noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in the equation (2) diverges. So, whenever $(a;q)_\infty$ is involved in a given formula, the constraint $|q|<1$ will be tacitly assumed.

The following notations are also frequently used in our investigation:

$$(a_1, a_2, \cdots, a_m; q)_n := (a_1;q)_n\,(a_2;q)_n\,\cdots\,(a_m;q)_n \tag{3}$$

and

$$(a_1, a_2, \cdots, a_m; q)_\infty := (a_1;q)_\infty\,(a_2;q)_\infty\,\cdots\,(a_m;q)_\infty. \tag{4}$$

Ramanujan (see [11] and [12]) defined the general theta function $\mathfrak{f}(a,b)$ as follows (see, for details, [5, p. 31, Eq. (18.1)] and [4]; see also [1], [15] and [22]):

$$\begin{aligned} f(a,b) &= 1 + \sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}(a^n+b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b,a) \qquad (|ab|<1). \end{aligned} \tag{5}$$

We find from this last equation (5) that

$$f(a,b) = a^{\frac{n(n+1)}{2}}\, b^{\frac{n(n-1)}{2}}\, f(a(ab)^n, b(ab)^{-n}) = f(b,a) \qquad (n \in \mathbb{Z}). \tag{6}$$

In fact, Ramanujan (see [11] and [12]) also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, is given by (see [5, p. 35, Entry 19]):

$$f(a,b) = (-a;ab)_\infty\,(-b;ab)_\infty\,(ab;ab)_\infty \tag{7}$$

or, equivalently, by (see [10])

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{n^2} z^n &= \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+zq^{2n-1}\right)\left(1+\frac{1}{z}\,q^{2n-1}\right) \\ &= \left(q^2;q^2\right)_\infty \left(-zq;q^2\right)_\infty \left(-\frac{q}{z};q^2\right)_\infty \qquad (|q|<1;\ z \neq 0). \end{aligned}$$

The q-series identity (7) or its above-mentioned equivalent form was first proved by Carl Friedrich Gauss (1777–1855).

Several q-series identities, which emerge naturally from Jacobi's triple-product identity (7), are worthy of note here (see, for details, [5, pp. 36–37, Entry 22]):

$$\begin{aligned} \varphi(q) &:= \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \\ &= \left\{(-q;q^2)_\infty\right\}^2 (q^2;q^2)_\infty = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}; \end{aligned} \tag{8}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}; \tag{9}$$

and

$$\mathfrak{f}(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q;q)_\infty. \tag{10}$$

Equation (10) is known as Euler's *Pentagonal Number Theorem*, which states that the number of partitions of a given positive integer n into distinct parts is equal to the number of partitions of n into odd parts. Remarkably, the following q-series identity:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{1}{\chi(-q)} \tag{11}$$

provides the analytic equivalence of Euler's famous theorem (see, for details, [3] and [8]).

We now recall the *first* Heine's transformation formula as follows (see [2, p. 19]):

$$\sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(c;q)_n\,(q;q)_n}\, z^n = \frac{(a;q)_\infty\,(bt;q)_\infty}{(c;q)_\infty\,(z;q)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{c}{a};q\right)_n\,(z;q)_n}{(bz;q)_n\,(q;q)_n}\, a^n,$$

which, upon first replacing z by $-\frac{qz}{b}$ and then letting $b \to \infty$ and $c \to 0$, yields

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, q^{\frac{n(n+1)}{2}}\, z^n = (a;q)_\infty\,(-qz;q)_\infty \sum_{n=0}^{\infty} \frac{a^n}{(-qz;q)_n\,(q;q)_n}. \tag{12}$$

On the other hand, the *second* Heine's transformation formula is recalled here as follows (see [2, p. 39]):

$$\sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(c;q)_n\,(q;q)_n}\, z^n = \frac{\left(\frac{c}{b};q\right)_\infty\,(bz;q)_\infty}{(c;q)_\infty\,(z;q)_\infty} \sum_{n=0}^{\infty} \frac{(\frac{abz}{c};q)_n\,(b;q)_n}{(bz;q)_n\,(q;q)_n} \left(\frac{c}{b}\right)^n.$$

Just as in the above-mentioned derivation of the equation (12), upon first replacing z by $-\frac{qz}{b}$ and then letting $b \to \infty$ and $c \to 0$, we find that

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, q^{\frac{n(n+1)}{2}}\, z^n = (-qz;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}\,(-az)^n}{(-qz;q)_n\,(q;q)_n}. \tag{13}$$

Furthermore, the q-binomial theorem is given by

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad (|z| < 1), \tag{14}$$

which was discovered by Cauchy [7].

2. Main Results

In this section, we establish a set of two presumably new theta-function identities which are based upon the q-binomial theorem (14) as well as Heine's transformations (12) and (13).

Theorem 2.1. *If*

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

then

$$1 + \left(\frac{1-a}{1+q}\right) q + \left(\frac{(1-a)(1-aq^2)}{(1+a)(1+q^2)}\right) q^2 + \cdots$$

$$+\varphi(-q)\left[1+\left(\frac{1-a}{1-q}\right)q+\left(\frac{(1-a)(1-aq^2)}{(q)_2}\right)q^2+\cdots\right]$$
$$=2(1-a\,q^{1^2}+a^2\,q^{2^2}-a^3\,q^{3^2}+\cdots) \tag{15}$$

and

$$\sum_{n=0}^{\infty}(-1)^n\,\frac{q^{\frac{n(n+3)}{2}}}{(q;q)_n\,(1+q^{n+1})^2}=\frac{\varphi(-q)}{q}\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}, \tag{16}$$

provided that each member of the assertions (15) *and* (16) *exists.*

Proof. In order to prove the assertion (15), we make use of such results as the q-binomial theorem (14) and Heine's transformations (12) and (13), and also of the explicit evaluations of the functions $\varphi(q)$ and $\psi(q)$ in the equations (8) and (9), respectively. First of all, we consider left-hand side of the equation (15) and we have

$$\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(-q;q)_n}\,q^n+\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(q;q)_n}\,q^n$$
$$=\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(q^2;q^2)_n}\,(q;q)_n\,q^n+\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(q^2;q^2)_n}\,(-q;q)_n\,q^n$$
$$=(q;q)_\infty\left(\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(q^2;q^2)_n}\,\frac{q^n}{(q^{n+1};q)_\infty}+\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(q^2;q^2)_n}\,\frac{q^n}{(-q^{n+1};q)_\infty}\right)$$
$$=(q;q)_\infty\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[1+(-1)^m]\,\frac{(a;q^2)_n}{(q^2;q^2)_n\,(q;q)_m}\,q^{n+m(n+1)}, \tag{17}$$

which, in view of (12), (13) and (14), readily yields

$$\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(-q;q)_n}\,q^n+\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(q;q)_n}\,q^n$$
$$=2(q;q)_\infty\sum_{m=0}^{\infty}\frac{q^{2m}}{(q;q)_{2m}}\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(q^2;q^2)_n}\,q^{(2m+1)n}$$
$$=2(q;q)_\infty\sum_{m=0}^{\infty}\frac{(aq^{2m+1};q^2)_\infty}{(q;q)_{2m}(q^{2m+1};q^2)_\infty}\,q^{2m}$$
$$=2(q^2;q^2)_\infty\,(aq;q^2)_\infty\sum_{m=0}^{\infty}\frac{q^{2m}}{(q^2;q^2)_m\,(aq;q^2)_m}$$
$$=2\sum_{n=0}^{\infty}(-a)^n\,q^{n^2}$$
$$=2(1-a\,q^{1^2}+a^2\,q^{2^2}-a^3\,q^{3^2}+\cdots). \tag{18}$$

This last member of the equation (18) is precisely the right-hand side of (15). Hence we have established the first assertion (15) of the Theorem.

The analogous proof of the second assertion (16) may be left as an exercise for the interested readers. We thus have completed our proof of the above Theorem. □

3. An Open Problem

Based upon the work presented in this paper, we find it to be worthwhile to motivate the interested reader to consider the following related open problem.

Open Problem. Find new analogous or more general identities and their possible applications in theoretical or applied sciences?

4. Concluding Remarks and Observations

The present investigation was motivated by several recent developments dealing essentially with theta-function identities and combinatorial partition-theoretic identities. We have established a set of two presumably new theta-function identities which are based upon the q-binomial theorem and Heine's transformations. We have also considered several closely-related identities such as (for example) q-product identities and Jacobi's triple-product identity.

In a recently-published review-cum-expository review article, in addition to applying the q-analysis to Geometric Function Theory of Complex Analysis, Srivastava [14] pointed out the fact that the results for the q-analogues can easily (and possibly trivially) be translated into the corresponding results for the (p, q)-analogues (with $0 < |q| < p \leqq 1$) by applying some obvious parametric and argument variations, the additional parameter p being *redundant*. Of course, this exposition and observation of Srivastava [14, p. 340] would apply also to the results which we have considered in our present investigation for $|q| < 1$.

Finally, with a view to further motivating researches involving theta-function identities and combinatorial partition-theoretic identities, we have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article. The list of citations, which we have included in this article, is believed to be potentially useful for indicating some of the directions for further researches and related developments on the subject-matter which we have dealt with here. In particular, we have cited the recent works by Cao *et al.* [6] and Srivastava *et al.* (see [16] to [18]; see also [21]).

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$$1 + \frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}\, x + \frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{\gamma})(1-q^{\gamma+1})}\, x^2 + \cdots ,$$

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