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On The Weighted Variable Exponent Lorentz Spaces

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Abstract

In this paper, using measure *wdm* instead of Haar measure *m*, weighted variable exponent Lorentz space is introduced and investigated. Then Hölder inequality is proved for weighted variable exponent Lorentz spaces. Also boundedness of the bilinear Hardy-Littlewood maximal function and Littlewood-Paley square function is considered on these spaces

Keywords: Weighted variable exponent Lorentz space, Bilinear Hardy-Littlewood maximal, Bilinear Littlewood-Paley square function

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1. Introduction

Throughout this paper, a continuous function ω satisfying $1 \leq \omega(x)$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for $x, y \in \mathbb{R}$ will be called a weight function on $\mathbb{R}$. We use the notation $\omega_1 \prec \omega_2$ to mean that there is a positive constant C such that $\omega_1(x) \leq C\omega_2(x)$ for all $x \in \mathbb{R}$. For $1 \leq p \leq \infty$, we write $L^p_\omega(\mathbb{R})$ as the usual weighted Lebesgue spaces. Moreover, the space $L^1_{loc}(\mathbb{R})$ consists of all (equivalence classes) measurable functions f on $\mathbb{R}$ such that $f\chi_K \in L^1(\mathbb{R})$ for every compact subset $K \subset \mathbb{R}$, where χ_K is the characteristic function of K. Let μ be a Borel measure on $\mathbb{R}$. The distribution function of f is defined as

$$\lambda^\omega_f(y) = \omega(\{x \in \mathbb{R} : |f(x)| > y\}) = \int\limits_{\{x\in\mathbb{R}:|f(x)|>y\}} \omega(x)\, d\mu(x).$$

The rearrangement function of f is given by

$$f^*_\omega(t) = \inf\left\{y > 0 : \lambda^\omega_f(y) \leq t\right\} = \sup\left\{y > 0 : \lambda^\omega_f(y) > t\right\}, t \geq 0.$$

Also, the average function of f is defined to be

$$f^{**}_\omega(t) = \frac{1}{t}\int\limits_0^t f^*_\omega(s)\, ds.$$

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If we take $\omega = 1$, then we use the notations f^*, f^{**}, λ_f instead of f_1^*, f_1^{**}, λ_f^1. Let $0 < l \leq \infty$. We denote

$$p_- = \inf_{x \in [0,l]} p(x), \qquad p^+ = \sup_{x \in [0,l]} p(x).$$

Moreover, we use the notation

$$P_a = \{p : a < p_- \leq p^+ < \infty\}, \; a \in \mathbb{R}.$$

In this paper, we are interested in the special cases of the P_a with $a = 0$ or $a = 1$. The set $\wp[0,l]$ is the family of $p \in L^\infty([0,l])$ such that there exist the limits $p(0) = \lim_{x \to 0} p(x)$, $p(\infty) = \lim_{x \to \infty} p(x)$ and the conditions

$$|p(x) - p(0)| \leq \frac{C}{\ln \frac{1}{|x|}}, \quad |x| \leq \frac{1}{2} \;\; (C > 0)$$

and

$$|p(x) - p(\infty)| \leq \frac{C}{\ln(e + |x|)}, \quad (C > 0) \tag{1.1}$$

are satisfied. If $l = \infty$, then it's enough to the inequality (1.1) satisfies. We also denote $\wp_a([0,l]) = \wp([0,l]) \cap P_a([0,l])$. Let Ω be an open set in $\mathbb{R}$. We denote $l = \mu(\Omega)$. Assume that $p, q \in \wp_0([0,l])$. The variable exponent Lorentz space $L^{p(.),q(.)}(\Omega)$ is defined as the set of all (equivalence classes) measurable functions f on Ω such that $\rho_{p,q}(f) < \infty$, where

$$\rho_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)} - 1} (f^*(t))^{q(t)} \, dt. \tag{1.2}$$

We use the notation

$$\|f\|^1_{L^{p(.),q(.)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p,q}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

Let $p \in \wp_0([0,l])$ and $q \in \wp_1([0,l])$. If $l = \infty$, then the equality (1.2) is equivalent to the following sum

$$\int_0^1 t^{\frac{q(0)}{p(0)} - 1} (f^*(t))^{q(t)} \, dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)} - 1} (f^*(t))^{q(t)} \, dt.$$

If $l < \infty$, then the equality (1.2) is equivalent to the integral

$$\int_0^l t^{\frac{q(0)}{p(0)} - 1} (f^*(t))^{q(t)} \, dt.$$

The space $L^{p(.),q(.)}(\Omega)$ is a normed vector space with norm

$$\|f\|_{L^{p(.),q(.)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p,q}\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

where

$$\rho_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)} - 1} (f^{**}(t))^{q(t)} \, dt$$

It is well known that $\|f\|^1_{L^{p(.),q(.)}(\Omega)} \leq \|f\|_{L^{p(.),q(.)}(\Omega)}$ for all $f \in L^{p(.),q(.)}(\Omega)$ [5]. In recent times, variable exponent Lebesgue and Lorentz spaces have attracted interest. Also these spaces have an important place in operator theory. Many researchers consider properties of these spaces and boundedness of some operators [3, 4, 5, 9, 10, 11, 12, 13, 14]. We briefly summarize some recent works in the literature. Ephremidze and Kokilashvili [5] considered the variable exponent Lorentz spaces and proved the boundedness of maximal, singular and fractional type operators on these spaces. Then, Kempka and Vybiral [7] introduced a new variable exponent Lorentz space and investigated properties of including, embedding, etc. Moreover, they proved the boundedness of singular and fractional type operators on new variable exponent Lorentz spaces. In [8], Kulak considered the inclusion theorems of these spaces by using measures. On the other hand, various results on boundedness in Lebesgue spaces were obtained for the bilinear Hardy-Littlewood maximal function and the bilinear Littlewood-Paley square function [3, 11, 12, 13, 14, 15]. Furthermore, the boundedness of bilinear Littlewood-Paley square function on variable exponent Lorentz spaces was proved in the work of Kulak [9]. In present paper, a new weighted variable exponent Lorentz space is introduced by using measure *wdm* instead of Haar measure *m*. Then, it is considered boundedness of the bilinear Littlewood-Paley square function and Hardy-Littlewood maximum function on weighted variable exponent Lorentz spaces under some conditions. Also using properties of Hardy-Littlewood maximal function, generalized Hölder inequality is proved for weighted variable exponent Lorentz spaces.

2. Main Results

2.1. The Weighted Variable Exponent Lorentz Spaces

Definition 2.1. Let $\Omega \subset \mathbb{R}$ and $\mu = \omega(x)\,dx$. Denote by $l = \mu(\Omega)$. Assume that $p, q \in \wp_0([0,l])$ and ω is a weight function. The weighted variable exponent Lorentz space $L_\omega^{p(.),q(.)}(\Omega)$ is the collection of all the (equivalence classes) measurable functions f such that $\|f\|^1_{L_\omega^{p(.),q(.)}(\Omega)} < \infty$ where

$$\|f\|^1_{L_\omega^{p(.),q(.)}(\Omega)} = \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_\omega^*(t) \right\|_{L^{q(.)}([0,l])} < \infty$$

and $L^{q(.)}([0,l])$ is a variable exponent Lebesgue space. Also, the space $L_\omega^{p(.),q(.)}(\Omega)$ is a normed space with the norm $\|.\|_{L_\omega^{p(.),q(.)}(\Omega)}$ defined by

$$\|f\|_{L_\omega^{p(.),q(.)}(\Omega)} = \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_\omega^{**}(t) \right\|_{L^{q(.)}([0,l])}.$$

Especially if we take $\omega = 1$, then we obtain the variable exponent Lorentz spaces.

Assume that $\mu = \omega(x)\,dx$ in Definition 2.3 in [8]. Then, the following theorem is obtained.

Theorem 2.2. *Let $p, q \in \wp_1([0,l])$. Then following properties are satisfied:* ***a)*** $\|f\|^1_{L_\omega^{p(.),q(.)}(\Omega)} \simeq \|f\|_{L_\omega^{p(.),q(.)}(\Omega)}$*, for all* $f \in L_\omega^{p(.),q(.)}(\Omega)$*,*
b) $L_\omega^{p(.),q(.)}(\Omega)$ *is a Banach function space.*

Lemma 2.3. *Let $p, q \in \wp_1([0,l])$. The set $\mathfrak{I}$ is dense in $L_\omega^{p(.),q(.)}(\Omega)$, where $\mathfrak{I}$ is set of simple integrable functions.*

Proof. Take any $f \in \mathfrak{I}$. Assume that $f(x) = \sum_{j=1}^{N} c_j \chi_{E_j}(x)$ where $c_1 > c_2 > ... > c_N > 0$ and E_j is a disjoint set with finite measure $(j = 1, 2, ..., N)$. Then, we know that

$$f_\omega^*(t) = \left\{ \begin{array}{ll} c_1, & 0 < t < a_1 \\ c_k, & a_{k-1} \le t < a_k \\ 0, & t \ge a_N \end{array} \right. ,$$

where $a_0 = 0$, $a_1 = \mu(E_1)$, $a_2 = \mu(E_1) + \mu(E_2)$,...,$a_{N-1} = \sum_{j=1}^{N-1} \mu\left(E_j\right)$, $a_N = \sum_{j=1}^{N} \mu\left(E_j\right)$ [6]. Firstly, suppose that

$l = \mu(\Omega) < \infty$. Then we have

$$
\begin{aligned}
\rho^{\omega}_{p,q}(f) &\simeq \int_0^l t^{\frac{q(0)}{p(0)}-1} \left(f^*_\omega(t)\right)^{q(t)} dt \\
&= \int_{[0,l]\cap[a_0,a_1)} t^{\frac{q(0)}{p(0)}-1} c_1^{q(t)} dt + \int_{[0,l]\cap[a_1,a_2)} t^{\frac{q(0)}{p(0)}-1} c_2^{q(t)} dt + \\
&\quad ...+ \int_{[0,l]\cap[a_{N-1},a_N)} t^{\frac{q(0)}{p(0)}-1} c_N^{q(t)} dt \\
&\le \int_{a_0}^{a_1} (c_1+1)^{q^+}\, t^{\frac{q(0)}{p(0)}-1} dt + \int_{a_1}^{a_2} (c_2+1)^{q^+}\, t^{\frac{q(0)}{p(0)}-1} dt + \\
&\quad ...+ \int_{a_{N-1}}^{a_N} (c_N+1)^{q^+}\, t^{\frac{q(0)}{p(0)}-1} dt \\
&= (c_1+1)^{q^+} \frac{p(0)}{q(0)} \left(a_1^{\frac{q(0)}{p(0)}} - a_0^{\frac{q(0)}{p(0)}}\right) + (c_2+1)^{q^+} \frac{p(0)}{q(0)} \left(a_2^{\frac{q(0)}{p(0)}} - a_1^{\frac{q(0)}{p(0)}}\right) + \\
&\quad ...+ (c_N+1)^{q^+} \frac{p(0)}{q(0)} \left(a_N^{\frac{q(0)}{p(0)}} - a_{N-1}^{\frac{q(0)}{p(0)}}\right) < \infty.
\end{aligned}
$$

So we obtain $f \in L^{p(.),q(.)}_\omega(\Omega)$. Now let $l = \mu(\Omega) = \infty$. Then by using similar technique, we find that

$$
\begin{aligned}
\rho^{\omega}_{p,q}(f) &\simeq \int_0^1 t^{\frac{q(0)}{p(0)}-1} \left(f^*_\omega(t)\right)^{q(t)} dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1} \left(f^*_\omega(t)\right)^{q(t)} dt \\
&\le (c_1+1)^{q^+} \frac{p(0)}{q(0)} \left(a_1^{\frac{q(0)}{p(0)}} - a_0^{\frac{q(0)}{p(0)}}\right) + (c_2+1)^{q^+} \frac{p(0)}{q(0)} \left(a_2^{\frac{q(0)}{p(0)}} - a_1^{\frac{q(0)}{p(0)}}\right) + \\
&\quad ...+ (c_N+1)^{q^+} \frac{p(0)}{q(0)} \left(a_N^{\frac{q(0)}{p(0)}} - a_{N-1}^{\frac{q(0)}{p(0)}}\right) + \\
&\quad + (c_1+1)^{q^+} \frac{p(\infty)}{q(\infty)} \left(a_1^{\frac{q(\infty)}{p(\infty)}} - a_0^{\frac{q(\infty)}{p(\infty)}}\right) + (c_2+1)^{q^+} \frac{p(\infty)}{q(\infty)} \left(a_2^{\frac{q(\infty)}{p(\infty)}} - a_1^{\frac{q(\infty)}{p(\infty)}}\right) + \\
&\quad ...+ (c_N+1)^{q^+} \frac{p(\infty)}{q(\infty)} \left(a_N^{\frac{q(\infty)}{p(\infty)}} - a_{N-1}^{\frac{q(\infty)}{p(\infty)}}\right) < \infty.
\end{aligned}
$$

Hence we have $f \in L^{p(.),q(.)}_\omega(\Omega)$. That means $\mathfrak{I} \subset L^{p(.),q(.)}_\omega(\Omega)$. Now take any $f \ge 0 \in L^{p(.),q(.)}_\omega(\Omega)$. There exists $(f_n)_{n\in\mathbb{N}} \subset \mathfrak{I}$ such that $f_n \nearrow f$, (a.e). Then $(f_n - f) \nearrow 0$, (a.e). Also since $L^{p(.),q(.)}_\omega(\Omega)$ is a Banach function space, we have $\|f_n - f\|_{L^{p(.),q(.)}_\omega(\Omega)} \nearrow 0$. Therefore we obtain that $\overline{\mathfrak{I}} = L^{p(.),q(.)}_\omega(\Omega)$. $\square$

Lemma 2.4. *Let $p, q \in \wp_1([0,\infty))$. Then*
a) *$C_c(\Omega)$ is dense in $L^{p(.),q(.)}_\omega(\Omega)$, where $C_c(\Omega)$ is the space of all continuous, complex-valued functions with compact support on Ω.*
b) *$C^\infty_c(\Omega)$ is dense in $L^{p(.),q(.)}_\omega(\Omega)$, where $C^\infty_c(\Omega)$ denotes the space of infinitely differentiable complex-valued functions with compact support on Ω.*

Proof. **a)** Let $f \in C_c(\Omega)$. If we set $supp f = K$ where K is a compact subset, then there exists $M > 0$ such that

$\max_{x\in K}|f(x)| = M$. Since $\mu(K) < \infty$, we have

$$\begin{aligned}\rho_{p,q}(f) &= \int_0^l t^{\frac{q(t)}{p(t)}-1}(f_\omega^*(t))^{q(t)}\,dt \simeq \int_0^{\mu(K)} t^{\frac{q(0)}{p(0)}-1}(f_\omega^*(t))^{q(t)}\,dt \\ &\le \int_0^{\mu(K)} t^{\frac{q(0)}{p(0)}-1}(M)^{q(t)}\,dt \le (M+1)^{q^+}\int_0^{\mu(K)} t^{\frac{q(0)}{p(0)}-1}dt \\ &= \frac{p(0)}{q(0)}(M+1)\mu(K)^{\frac{q(0)}{p(0)}} < \infty.\end{aligned}$$

That means $f \in L_\omega^{p(.),q(.)}(\Omega)$. So we write $C_c(\Omega) \subset L_\omega^{p(.),q(.)}(\Omega)$. Now, let $f \in L_\omega^{p(.),q(.)}(\Omega)$. For given $0 < \varepsilon < 1$, there exists $h \in \mathfrak{I}$ such that

$$\|f-h\|_{L_\omega^{p(.),q(.)}(\Omega)} < \frac{\varepsilon}{2} \tag{2.1}$$

by Lemma 2.3. On the other hand, by Lusin theorem [16] there exists $g \in C_c(\Omega)$ such that $\mu(A) < \left(\frac{\varepsilon}{4\frac{p(0)}{q(0)}(\|s\|_\infty+1)^{q^+}}\right)^{\frac{p(0)}{q(0)}}$ and $|g| < \|s\|_\infty$ where $A = \{x \in \Omega : g(x) \ne s(x)\}$. Then, we have

$$\|g-s\|_{L_\omega^{p(.),q(.)}(\Omega)} = \|g-s\|_{L_\omega^{p(.),q(.)}(A)} \le \|g\|_{L_\omega^{p(.),q(.)}(A)} + \|s\|_{L_\omega^{p(.),q(.)}(A)}. \tag{2.2}$$

Also since $\mu(A) < \infty$, we deduce that

$$\begin{aligned}\rho_{p,q}(g) &\simeq \int_0^{\mu(A)} t^{\frac{q(0)}{p(0)}-1}(g_\omega^*(t))^{q(t)}\,dt \le \int_0^{\mu(A)} t^{\frac{q(0)}{p(0)}-1}(\|s\|_\infty+1)^{q^+}\,dt \\ &= \frac{p(0)}{q(0)}(\|s\|_\infty+1)\mu(A)^{\frac{q(0)}{p(0)}}\end{aligned}$$

and so

$$\|g\|_{L_\omega^{p(.),q(.)}(A)} \le \frac{p(0)}{q(0)}(\|s\|_\infty+1)\mu(A)^{\frac{q(0)}{p(0)}}. \tag{2.3}$$

Similarly, we find that

$$\|s\|_{L_\omega^{p(.),q(.)}(A)} \le \frac{p(0)}{q(0)}(\|s\|_\infty+1)\mu(A)^{\frac{q(0)}{p(0)}}. \tag{2.4}$$

If we combine the inequalities (2.2), (2.3) and (2.4), then we have

$$\begin{aligned}\|g-s\|_{L_\omega^{p(.),q(.)}(\Omega)} &\le 2\frac{p(0)}{q(0)}(\|s\|_\infty+1)\mu(A)^{\frac{q(0)}{p(0)}} \\ &< 2\frac{p(0)}{q(0)}(\|s\|_\infty+1)\left(\left(\frac{\varepsilon}{4\frac{p(0)}{q(0)}(\|s\|_\infty+1)^{q^+}}\right)^{\frac{p(0)}{q(0)}}\right)^{\frac{q(0)}{p(0)}} = \frac{\varepsilon}{2}\end{aligned} \tag{2.5}$$

Finally using the inequalities (2.1) and (2.5), it follows that

$$\|f-g\|_{L_\omega^{p(.),q(.)}(\Omega)} \le \|f-h\|_{L_\omega^{p(.),q(.)}(\Omega)} + \|h-g\|_{L_\omega^{p(.),q(.)}(\Omega)} < \varepsilon.$$

So, the proof is completed.

b) It's known that $C_c^\infty(\Omega) \subset C_c(\Omega)$. Also by (a), we have $C_c^\infty(\Omega) \subset L_\omega^{p(.),q(.)}(\Omega)$. Now take any $f \in L_\omega^{p(.),q(.)}(\Omega)$. Since $C_c(\Omega)$ is dense in $L_\omega^{p(.),q(.)}(\Omega)$, there exists $g \in C_c(\Omega)$ such that

$$\|f-g\|_{L_\omega^{p(.),q(.)}(\Omega)} < \frac{\varepsilon}{1+C} \tag{2.6}$$

for all $\varepsilon > 0$. Also by from [18], for given $0 < \varepsilon < 1$, there exists $h \in C_c^\infty(\Omega)$ such that

$$\|g - h\|_\infty < \frac{\varepsilon}{1 + C}$$

where $C = \frac{p(0)}{q(0)}\mu(K)^{\frac{q(0)}{p(0)}}$ and $\operatorname{supp}(g - h) = K$. On the other hand, since $\frac{\varepsilon}{1+C} < 1$ and $1 \leq q(t)$, we have that

$$\begin{aligned}
\rho_{p,q}(g-h) &= \int_0^l t^{\frac{q(t)}{p(t)}-1}\left((g-h)_\omega^*(t)\right)^{q(t)} dt \simeq \int_0^{\mu(K)} t^{\frac{q(0)}{p(0)}-1}\left((g-h)_\omega^*(t)\right)^{q(t)} dt \\
&\leq \int_0^{\mu(K)} t^{\frac{q(0)}{p(0)}-1}\left(\|g-h\|_\infty\right)^{q(t)} dt < \int_0^{\mu(K)} t^{\frac{q(0)}{p(0)}-1}\left(\frac{\varepsilon}{1+C}\right) dt \\
&= \left(\frac{\varepsilon}{1+C}\right)\frac{p(0)}{q(0)}\mu(K)^{\frac{q(0)}{p(0)}} .
\end{aligned}$$

So, the last inequality implies that

$$\|g-h\|_{L_\omega^{p(.),q(.)}(\Omega)} < \left(\frac{\varepsilon}{1+C}\right)\frac{p(0)}{q(0)}\mu(K)^{\frac{q(0)}{p(0)}} . \tag{2.7}$$

Finally combining the inequalities (2.6) and (2.7), we obtain that

$$\begin{aligned}
\|f-h\|_{L_\omega^{p(.),q(.)}(\Omega)} &\leq \|f-g\|_{L_\omega^{p(.),q(.)}(\Omega)} + \|g-h\|_{L_\omega^{p(.),q(.)}(\Omega)} \\
&< \frac{\varepsilon}{1+C} + \left(\frac{\varepsilon}{1+C}\right)\frac{p(0)}{q(0)}\mu(K)^{\frac{q(0)}{p(0)}} \\
&= \frac{\varepsilon}{1+C}\left(1 + \frac{p(0)}{q(0)}\mu(K)^{\frac{q(0)}{p(0)}}\right) \\
&= \frac{\varepsilon}{1+C}(1+C) = \varepsilon.
\end{aligned}$$

□

If we take $\mu = \omega_1(x)\,dx$ and $\nu = \omega_2(x)\,dx$ in Theorem 1 and Lemma 3 in [8], the following theorem is given.

Theorem 2.5. ***a)*** *Let $l = \mu(\Omega) = \infty$, $p, q \in \wp_1([0, l])$, $q(0) < p(0)$ and $q(\infty) > p(\infty)$. Assume that ω_1 and ω_2 are two weight functions on Ω. Then, the inclusion*

$$L_{\omega_1}^{p(.),q(.)}(\Omega) \subset L_{\omega_2}^{p(.),q(.)}(\Omega)$$

holds if and only if there exists $C > 0$ such that

$$\|f\|_{L_{\omega_2}^{p(.),q(.)}(\Omega)} \leq C\,\|f\|_{L_{\omega_1}^{p(.),q(.)}(\Omega)}$$

for all $f \in L_{\omega_1}^{p(.),q(.)}(\Omega)$.
b) *Let $l = \mu(\Omega) < \infty$, $p, q \in \wp_1([0, l])$. Assume that ω_1 and ω_2 are two weight functions on Ω. Then, the inclusion*

$$L_{\omega_1}^{p(.),q(.)}(\Omega) \subset L_{\omega_2}^{p(.),q(.)}(\Omega)$$

holds if and only if there exists $C > 0$ such that

$$\|f\|_{L_{\omega_2}^{p(.),q(.)}(\Omega)} \leq C\,\|f\|_{L_{\omega_1}^{p(.),q(.)}(\Omega)}$$

for all $f \in L_{\omega_1}^{p(.),q(.)}(\Omega)$.
c) *If $\omega_1 \prec \omega_2$, then the inequality*

$$\|f\|_{L_{\omega_1}^{p(.),q(.)}(\Omega)} \leq C\,\|f\|_{L_{\omega_2}^{p(.),q(.)}(\Omega)}, \ (C > 0)$$

holds for all $f \in L_{\omega_2}^{p(.),q(.)}(\Omega)$.

Lemma 2.6. ***Hölder inequality for weighted variable exponent Lorentz spaces*** *[8] Let* $1 \leq q(.) \leq q^+ < \infty$, $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$ *and* $\frac{1}{q(.)} + \frac{1}{q'(.)} = 1$. *If* $f \in L_\omega^{p(.),q(.)}(\Omega)$ *and* $g \in L_\omega^{p'(.),q'(.)}(\Omega)$ *then* $fg \in L^1(\Omega)$ *and there exists* $C > 0$ *such that*

$$\int_\Omega |f(x) g(x)| \, dx \leq C \|f\|_{L_\omega^{p(.),q(.)}(\Omega)} \|g\|_{L_\omega^{p'(.),q'(.)}(\Omega)}.$$

Theorem 2.7. ***a)*** *Let* $p \in \wp_1([0,l])$ *and* $1 \leq q < \infty$. *If* $p(\infty) \leq q$, $q \leq p(0)$, $\frac{1}{q} + \frac{1}{q'} = 1$ *and* $\frac{1}{\omega^{\frac{1}{q}}} \in L^{q'}(\Omega)$, *then the following inclusion is satisfied:*

$$L_\omega^{p(.),q}(\Omega) \subset L^1(\Omega).$$

b) *Let* $p, q \in \wp_1([0,l])$ *and* $1 \leq q(.) \leq q^+ < \infty$. *Then the inclusion*

$$L_\omega^{p(.),q(.)}(\Omega) \subset L_{loc}^1(\Omega)$$

holds.
c) *Let* $p, q \in \wp_1([0,l])$, $\frac{1}{p(.)} + \frac{1}{p(.)'} = 1$ *and* $\frac{1}{q(.)} + \frac{1}{q(.)'} = 1$. *If there exists* $h \in L_\omega^{p'(.),q'(.)}(\Omega)$ *such that* $|h(x)| \geq 1$, *(a.e), then the inclusion*

$$L_\omega^{p(.),q(.)}(\Omega) \subset L^1(\Omega)$$

holds.

Proof. **a)** Take any $f \in L_\omega^{p(.),q}(\Omega)$. Assume that $l = \mu(\Omega) = \infty$. Then using assumptions, we have

$$\begin{aligned} \int_\Omega |f(x)|^q \omega(x) \, dx &= \int_0^l f_\omega^*(t)^q \, dt = \int_0^1 f_\omega^*(t)^q \, dt + \int_1^\infty f_\omega^*(t)^q \, dt \\ &\leq \int_0^1 t^{\frac{q}{p(0)}-1} f_\omega^*(t)^q \, dt + \int_1^\infty t^{\frac{q}{p(\infty)}-1} f_\omega^*(t)^q \, dt < \infty. \end{aligned}$$

So we find that $f \in L_\omega^q(\Omega)$. On the other hand, since $\frac{1}{\omega^{\frac{1}{q}}} \in L^{q'}(\Omega)$, we have

$$\int_\Omega |f(x)| \, dx = \int_\Omega |f(x)| \, \omega^{\frac{1}{q}}(x) \frac{1}{\omega^{\frac{1}{q}}(x)} dx \leq \|f\|_{L_\omega^q(\Omega)} \left\| \frac{1}{\omega^{\frac{1}{q}}} \right\|_{L^{q'}(\Omega)} < \infty.$$

That means $f \in L^1(\Omega)$. Hence, we obtain that $L_\omega^{p(.),q}(\Omega) \subset L_\omega^q(\Omega) \subset L^1(\Omega)$. Similarly, it's proved for $l < \infty$.
b) Assume that $f \in L_\omega^{p(.),q(.)}(\Omega)$. Let K be a compact subset. By the Theorem 2.2, we know that $L_\omega^{p'(.),q'(.)}(\Omega)$ is a Banach function space. Therefore, we can write $\chi_K \in L_\omega^{p'(.),q'(.)}(\Omega)$. So using Lemma 2.6, we have that

$$\int_\Omega |f(x)| \chi_K(x) \, dx \leq \|f\|_{L_\omega^{p(.),q(.)}(\Omega)} \|\chi_K\|_{L_\omega^{p'(.),q'(.)}(\Omega)} < \infty$$

where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$ and $\frac{1}{q(.)} + \frac{1}{q'(.)} = 1$. From this result, we write $f \in L_{loc}^1(\Omega)$. Hence we obtain that $L_\omega^{p(.),q(.)}(\Omega) \subset L_{loc}^1(\Omega)$.
c) Let $f \in L_\omega^{p(.),q(.)}(\Omega)$. Assume that there exists $h \in L_\omega^{p'(.),q'(.)}(\Omega)$ such that $|h(x)| \geq 1$, (a.e). So we write that

$$\int_\Omega |f(x)| \, dx \leq \int_\Omega |f(x) h(x)| \, dx \leq \|f\|_{L_\omega^{p(.),q(.)}(\Omega)} \|h\|_{L_\omega^{p'(.),q'(.)}(\Omega)} < \infty$$

and we have $f \in L^1(\Omega)$. Therefore the proof is completed. □

2.2. Boundedness of The Bilinear Hardy-Littlewood Maximal Function and The Bilinear Littlewood-Paley Square Function on Weighted Variable Exponent Lorentz Spaces

Definition 2.8. [2] The linear Hardy-Littlewood maximal function M is defined by

$$M(h)(x) = \sup_{t>0} \frac{1}{2t} \int_{-t}^{t} |h(x-y)|\, dy,\ x \in \mathbb{R}$$

for all $h \in L^1_{loc}(\mathbb{R})$.

Definition 2.9. [1, 4, 11] The bilinear Hardy-Littlewood maximal function M is defined by

$$M(f,g)(x) = \sup_{t>0} \frac{1}{2t} \int_{-t}^{t} |f(x+y)\, g(x-y)|\, dy,\ x \in \mathbb{R}$$

for all $f, g \in L^1_{loc}(\mathbb{R})$.

Firstly, we will show that the bilinear Hardy-Littlewood maximal function is unbounded under some conditions.

Theorem 2.10. ***a)*** *Let $p_1, q_1, p_2, q_2, p \in \wp_1([0,\infty])$, $1 \le q < \infty$ and ω_i $(i = 1, 2, 3)$ be weight functions. If $p(\infty) \le q$, $q \le p(0)$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{\omega_3^{\frac{1}{q}}} \in L^{q'}(\mathbb{R})$, then the bilinear Hardy-Littlewood maximal function M is not bounded from $L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R}) \times L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})$ to $L^{p(.),q}_{\omega_3}(\mathbb{R})$.*
b) *Let $p_3, q_3 \in \wp_1([0,\infty])$, $\frac{1}{p_3(.)} + \frac{1}{p_3(.)'} = 1$, $\frac{1}{q_3(.)} + \frac{1}{q_3(.)'} = 1$ and ω_i $(i = 1, 2, 3)$ be weight functions. If there exists $h \in L^{p_3'(.),q_3'(.)}_{\omega_3}(\mathbb{R})$ such that $|h(x)| \ge 1$, (a.e), then the bilinear Hardy-Littlewood maximal function M is not bounded from $L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R}) \times L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})$ to $L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})$.*

Proof. **a)** Since $p(\infty) \le q$, $q \le p(0)$ and $\frac{1}{\omega_3^{\frac{1}{q}}} \in L^{q'}(\mathbb{R})$, we write that $L^{p(.),q}_{\omega_3}(\mathbb{R}) \subset L^1(\mathbb{R})$ by Theorem 2.7. Take any indicator functions χ_A, χ_B where $A, B \subset \mathbb{R}$ are compact subsets. By Theorem 2.2, we have $\chi_A \in L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R})$ and $\chi_B \in L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})$. On the other hand, M is unbounded on $\left(L^1 \times L^1\right)(\mathbb{R})$ [1]. Since χ_A, $\chi_B \in L^1(\mathbb{R})$ are identically zero, then the function M is not integrable on $\mathbb{R}$. That means $M(\chi_A, \chi_B)$ is not in $L^1(\mathbb{R})$. So we have $M(\chi_A, \chi_B) \notin L^{p(.),q}_{\omega_3}(\mathbb{R})$. This completes the proof.
b) From the assumptions and Theorem 2.7, we have $L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R}) \subset L^1(\mathbb{R})$. Using similar technique in proof of (a), we obtain that $M(\chi_A, \chi_B) \notin L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})$. Therefore we say that M is not bounded from $L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R}) \times L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})$ to $L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})$. □

Now, we will consider boundedness of the bilinear Hardy-Littlewood maximal function and bilinear Littlewood-Paley Square Function.

Theorem 2.11. *Let $p_1, q_1, p_2, q_2, p_3, q_3 \in \wp_1([0,\infty])$ and ω_i $(i = 1, 2, 3)$ be weight functions. Assume that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\omega_3 \prec \omega_i$ $(i = 1, 2)$. Then the bilinear Hardy-Littlewood maximal function M is bounded from $L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R}) \times L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})$ to $L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})$. Furthermore there exists $C > 0$ such that*

$$\|M(f,g)\|_{L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})} \le C \|f\|_{L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R})} \|g\|_{L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})} .$$

Proof. Take any $f \in L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R})$ and $g \in L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})$. It's known that the inequality

$$M(f,g) \le C_1 M\left(f^{\frac{1}{\theta}}\right)^{\theta} M\left(g^{\frac{1}{1-\theta}}\right)^{1-\theta}$$

holds for every $0 < \theta < 1$ [4]. If we use this last inequality, then the inequality

$$(M(f,g))^*_{\omega_3}(s) \le C_1 \left(M\left(f^{\frac{1}{\theta}}\right)^*_{\omega_3}\right)^{\theta}\left(\frac{s}{2}\right)\left(M\left(g^{\frac{1}{1-\theta}}\right)^*_{\omega_3}\right)^{1-\theta}\left(\frac{s}{2}\right) \tag{2.8}$$

is obtained. Also we know that the linear maximal function satisfies the following inequality

$$M(f)^{*}_{\omega_3} \le f^{**}_{\omega_3} \tag{2.9}$$

[2]. So combining the inequalities (2.8) and (2.9), we have

$$\begin{aligned}(M(f,g))^{*}_{\omega_3}(s) &\le C_1\left(\left(f^{\frac{1}{\theta}}\right)^{**}_{\omega_3}\right)^{\theta}\left(\frac{s}{2}\right)\left(\left(g^{\frac{1}{1-\theta}}\right)^{**}_{\omega_3}\right)^{1-\theta}\left(\frac{s}{2}\right)\\ &= C_1\left((f)^{**}_{\omega_3}\right)^{\frac{1}{\theta}\theta}\left(\frac{s}{2}\right)\left((g)^{**}_{\omega_3}\right)^{\frac{1}{1-\theta}(1-\theta)}\left(\frac{s}{2}\right)\\ &= C_1 f^{**}_{\omega_3}\left(\frac{s}{2}\right)g^{**}_{\omega_3}\left(\frac{s}{2}\right).\end{aligned} \tag{2.10}$$

On the other hand, we find that

$$f^{**}_{\omega_3}\left(\frac{s}{2}\right) = \frac{1}{\frac{s}{2}}\int_0^{\frac{s}{2}} f^{*}_{\omega_3}(t)\,dt \le \frac{2}{s}\int_0^{s} f^{*}_{\omega_3}(t)\,dt = 2f^{**}_{\omega_3}(s). \tag{2.11}$$

Similarly, we deduce that

$$g^{**}_{\omega_3}\left(\frac{s}{2}\right) \le 2g^{**}_{\omega_3}(s). \tag{2.12}$$

From the inequalities (2.10), (2.11) and (2.12) , we find that

$$(M(f,g))^{*}_{\omega_3}(s) \le 4C_1 f^{**}_{\omega_3}(s)\, g^{**}_{\omega_3}(s). \tag{2.13}$$

So by using the inequality (2.13), (c) in Theorem 2.5 and generalized Hölder inequality for variable exponent Lebesgue spaces [17], we obtain that

$$\begin{aligned}\|M(f,g)\|_{L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})} &\simeq \|M(f,g)\|^{1}_{L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})} = \left\|t^{\frac{1}{p_3(t)}-\frac{1}{q_3(t)}}M(f,g)^{*}_{\omega_3}\right\|_{L^{q_3(.)}[0,\infty]}\\ &\le 4C_1\left\|t^{\frac{1}{p_3(t)}-\frac{1}{q_3(t)}}f^{**}_{\omega_3}(t)\,g^{**}_{\omega_3}(t)\right\|_{L^{q_3(.)}[0,\infty]}\end{aligned}$$

$$= 4C_1\left\|t^{\frac{1}{p_3(t)}-\frac{1}{q_3(t)}}t^{\left(\frac{1}{p_1(t)}-\frac{1}{q_1(t)}\right)}t^{-\left(\frac{1}{p_1(t)}-\frac{1}{q_1(t)}\right)}t^{\left(\frac{1}{p_2(t)}-\frac{1}{q_2(t)}\right)}t^{-\left(\frac{1}{p_2(t)}-\frac{1}{q_2(t)}\right)}f^{**}_{\omega_3}(t)\,g^{**}_{\omega_3}(t)\right\|_{L^{q_3(.)}[0,\infty]}$$

$$\begin{aligned}&= 4C_1\left\|\left(t^{\left(\frac{1}{p_1(t)}-\frac{1}{q_1(t)}\right)}f^{**}_{\omega_3}(t)\right)\left(t^{\left(\frac{1}{p_2(t)}-\frac{1}{q_2(t)}\right)}g^{**}_{\omega_3}(t)\right)\right\|_{L^{q_3(.)}[0,\infty]}\\ &\le 4C_1\left\|t^{\left(\frac{1}{p_1(t)}-\frac{1}{q_1(t)}\right)}f^{**}_{\omega_3}(t)\right\|_{L^{q_1(.)}[0,\infty]}\left\|\left(t^{\left(\frac{1}{p_2(t)}-\frac{1}{q_2(t)}\right)}g^{**}_{\omega_3}(t)\right)\right\|_{L^{q_2(.)}[0,\infty]}\\ &= 4C_1\|f\|_{L^{p_1(.),q_1(.)}_{\omega_3}(\mathbb{R})}\|g\|_{L^{p_2(.),q_2(.)}_{\omega_3}(\mathbb{R})} \le C\|f\|_{L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R})}\|g\|_{L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})}.\end{aligned}$$

This completes the proof. □

Proposition 2.12. ***Generalized Hölder inequality for weighted variable exponent Lorentz spaces*** *Let $p_1, q_1, p_2, q_2, p_3, q_3 \in \wp_1([0,\infty])$ and ω_i $(i = 1,2,3)$ be weight functions. Assume that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\omega_3 \prec \omega_i$ $(i = 1,2)$. If $f \in L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R})$ and $g \in L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})$, then $fg \in L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})$ and there exists $C > 0$ such that*

$$\|fg\|_{L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})} \le C\|f\|_{L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R})}\|g\|_{L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})}.$$

Proof. By from Corollary 3.6 in [2], we write that $|f(x)| \leq M(f)(x)$ and $|g(x)| \leq M(g)(x)$, (a.e). So we have $|f(x)g(x)| \leq M(f)(x)M(g)(x)$, (a.e). Using this last inequality and the inequalities (2.9), (2.11), (2.12), we have

$$\begin{aligned}(fg)^*_{\omega_3}(s) &\leq (M(f)M(g))^*_{\omega_3}(s) \leq M(f)^*_{\omega_3}\left(\frac{s}{2}\right)M(g)^*_{\omega_3}\left(\frac{s}{2}\right)\\ &\leq C_1 f^{**}_{\omega_3}\left(\frac{s}{2}\right)g^{**}_{\omega_3}\left(\frac{s}{2}\right) \leq 4C_1 f^{**}_{\omega_3}(s)\,g^{**}_{\omega_3}(s). \qquad (2.14)\end{aligned}$$

From the inequality (2.14), (c) in Theorem 2.5 and using similar technique in proof of Theorem 2.11, we obtain that

$$\begin{aligned}\|fg\|_{L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})} &\simeq \|fg\|^1_{L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})} = \left\|t^{\frac{1}{p_3(t)}-\frac{1}{q_3(t)}}(fg)^*_{\omega_3}\right\|_{L^{q_3(.)}[0,\infty]}\\ &\leq 4C_1\left\|t^{\frac{1}{p_3(t)}-\frac{1}{q_3(t)}}f^{**}_{\omega_3}g^{**}_{\omega_3}\right\|_{L^{q_3(.)}[0,\infty]}\\ &\leq 4C_1\left\|t^{\left(\frac{1}{p_1(t)}-\frac{1}{q_1(t)}\right)}f^{**}_{\omega_3}(t)\right\|_{L^{q_1(.)}[0,\infty]}\left\|\left(t^{\left(\frac{1}{p_2(t)}-\frac{1}{q_2(t)}\right)}g^{**}_{\omega_3}(t)\right)\right\|_{L^{q_2(.)}[0,\infty]}\\ &= 4C_1\|f\|_{L^{p_1(.),q_1(.)}_{\omega_3}(\mathbb{R})}\|g\|_{L^{p_2(.),q_2(.)}_{\omega_3}(\mathbb{R})}\\ &\leq C\|f\|_{L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R})}\|g\|_{L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})}.\end{aligned}$$

□

Definition 2.13. [11, 13] Let K be a smooth bump function defined on $\mathbb{R}$ such that $\hat{K}$ is supported in the unit interval of $\mathbb{R}$. Assume that K_n is the function defined by $\hat{K}_n(\xi) = \hat{K}(\xi - n)$ for $n \in \mathbb{Z}$. Define the bilinear operator

$$S_n(f,g)(x) = \int_{\mathbb{R}} f(x-y)\,g(x+y)\,K_n(y)\,dy$$

for $f, g \in C^\infty_c(\mathbb{R})$. Let $S(f,g)$ be the bilinear Littlewood-Paley square function associated with this sequence of operators, i.e.

$$S(f,g)(x) = \left(\sum_n |S_n(f,g)(x)|^2\right)^{\frac{1}{2}}.$$

Theorem 2.14. *Let $p_1, q_1, p_2, q_2, p_3, q_3 \in \wp_1([0,\infty])$ and $m \in S(\mathbb{R})$. For $n \in \mathbb{Z}$ define $m_n(\xi) = m(\xi - n)$ and let S_n be the bilinear multiplier operator associated with m_n and ω_i $(i = 1,2,3)$ be weight functions. Assume that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\omega_3 \prec \omega_i$ $(i = 1,2)$. Then the bilinear Littlewood-Paley square function $S(f,g)$ is bounded from $L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R}) \times L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})$ to $L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})$. Furthermore there exists $C > 0$ such that*

$$\|S(f,g)\|_{L^{p_3(.),q_3(.)}_{\omega_3}(\mathbb{R})} \leq C\|f\|_{L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R})}\|g\|_{L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})}$$

for all $f \in L^{p_1(.),q_1(.)}_{\omega_1}(\mathbb{R})$ and $g \in L^{p_2(.),q_2(.)}_{\omega_2}(\mathbb{R})$.

Proof. For $f, g \in C^\infty_c(\mathbb{R})$, the inequality $S(f,g)(x) \leq C_1\left(M\left(|f|^2, |g|^2\right)(x)\right)^{\frac{1}{2}}$, (a.e) is known from [13]. Take any

$f \in L_{\omega_1}^{p_1(.),q_1(.)}(\mathbb{R})$ and $g \in L_{\omega_2}^{p_2(.),q_2(.)}(\mathbb{R})$. Then using Theorem 2.11, we have

$$\begin{aligned}
\|S(f,g)\|_{L_{\omega_3}^{p_3(.),q_3(.)}(\mathbb{R})} &\leq C_1 \left\| \left(M\left(|f|^2, |g|^2 \right) \right)^{\frac{1}{2}} \right\|_{L_{\omega_3}^{p_3(.),q_3(.)}(\mathbb{R})} \\
&= C_1 \left\| t^{\frac{1}{p_3(t)} - \frac{1}{q_3(t)}} \left(M\left(|f|^2, |g|^2 \right)_{\omega_3}^{**} \right)^{\frac{1}{2}} \right\|_{L^{q_3(.)}[0,\infty]} \\
&= C_1 \left\| t^{\frac{1}{\frac{p_3(t)}{2}} - \frac{1}{\frac{q_3(t)}{2}}} M\left(|f|^2, |g|^2 \right)_{\omega_3}^{**} \right\|_{L^{\frac{q_3(.)}{2}}[0,\infty]} \\
&= C_1 \left\| M\left(|f|^2, |g|^2 \right) \right\|_{L_{\omega_3}^{\frac{p_3(.)}{2}, \frac{q_3(.)}{2}}(\mathbb{R})} \\
&\leq C_1 C_2 \left\| f^2 \right\|_{L_{\omega_1}^{\frac{p_1(.)}{2}, \frac{q_1(.)}{2}}(\mathbb{R})} \left\| g^2 \right\|_{L_{\omega_2}^{\frac{p_2(.)}{2}, \frac{q_2(.)}{2}}(\mathbb{R})} \\
&= C_1 C_2 \left\| t^{\frac{2}{p_1(.)} - \frac{2}{q_1(.)}} \left(f_{\omega_1}^{**} \right)^2 \right\|_{L^{\frac{q_1(.)}{2}}[0,\infty]} \left\| t^{\frac{2}{p_2(.)} - \frac{2}{q_2(.)}} \left(g_{\omega_2}^{**} \right)^2 \right\|_{L^{\frac{q_2(.)}{2}}[0,\infty]} \\
&= C_1 C_2 \left\| t^{\frac{1}{p_1(.)} - \frac{1}{q_1(.)}} f_{\omega_1}^{**} \right\|_{L^{q_1(.)}[0,\infty]} \left\| t^{\frac{1}{p_2(.)} - \frac{1}{q_2(.)}} g_{\omega_2}^{**} \right\|_{L^{q_2(.)}[0,\infty]} \\
&= C_3 \|f\|_{L_{\omega_1}^{p_1(.),q_1(.)}(\mathbb{R})} \|g\|_{L_{\omega_2}^{p_2(.),q_2(.)}(\mathbb{R})}.
\end{aligned}$$

□

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