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An Integral Formula Generated by Hurwitz-Lerch Zeta Function with Order 1

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Dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday

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Abstract

In this paper, we give some results associated with Hurwitz-Lerch zeta function with order 1 which is special case of Hurwitz-Lerch zeta function. One of the results is the integral formula including Hurwitz-Lerch zeta function with order 1. The other result is a corollary generated by integral representation of Hurwitz-Lerch zeta function with order 1.

Keywords: Hurwitz-Lerch zeta function, Hurwitz-Lerch zeta function with order 1, Riemann zeta function

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1. Introduction, definitions and preliminaries

Throughout this article, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Also,

$$\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\},$$

$$\mathbb{Z}_0 = \{-3, -2, -1, 0, 1, 2, 3, \cdots\} = \mathbb{Z} \cup \{0\}$$

and

$$\mathbb{Z}^- = \{-1, -2, -3, \cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

The Hurwitz-Lerch zeta function $\Phi(z, s, w)$ is defined by (cf. [3], [6], [7])

$$\Phi(z, s, w) = \sum_{n=0}^{\infty} \frac{z^n}{(n+w)^s} \tag{1.1}$$

$$(w \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ Re(s) > 1 \text{ when } |z| = 1).$$

For $z = 1$, $\Phi(1, s, w) = \zeta(s, w)$ is the Hurwitz zeta function (cf. [5], [10]) defined by

$$\zeta(s, w) = \sum_{n=0}^{\infty} \frac{1}{(n+w)^s}\ ,\ (Re(s) > 1,\ w \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

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For $z = 1$ and $w = 1$, $\Phi(1, s, 1) = \zeta(s)$ is the Riemann zeta function (cf. [5], [10]) defined by

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s} \ , (Re(s) > 1, w \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

For $w = 1$, $z\Phi(z, s, 1) = Li_s(z)$ is the polylogarithmic function (cf. [9]) defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \tag{1.2}$$

$$(s \in \mathbb{C} \text{ when } |z| < 1; Re(s) > 1 \text{ when } |z| = 1).$$

For $z = e^{2\pi in\xi}$ into (1.2), we arrive the Lerch zeta function (cf. [4]) defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1)$$

$$(\xi \in \mathbb{R}; Re(s) > 1).$$

Hence, we conclude that the Lerch zeta function, the polylogarithmic function, the Hurwitz zeta function, the Riemann zeta function are special cases of the Hurwitz-Lerch zeta function.

Also, Hurwitz-Lerch zeta function (cf. [6], [8]) has an integral representation as follows:

$$\Phi(z, s, w) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-wt}}{1 - ze^{-t}} dt$$

where

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt.$$

The Hurwitz-Lerch zeta function with order 1 (cf. [1], [2]) is defined by

$$\Phi(z, 1, w) = \sum_{n=0}^{\infty} \frac{z^n}{n + w} \ , (|z| < 1, w \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

In [1], Aygunes obtained some results associated to the Hurwitz-Lerch zeta function with order 1 as follows:

Theorem 1.1. *(cf. [1]) Let* $M \in \mathbb{N} \setminus \{0\}$ *and* $|z| < 1/2$. *Then, we have*

$$\frac{1}{1-z} \sum_{k=0}^{M-1} \binom{M-1}{k} \Phi\left(\frac{z}{1-z}, 1, k+1\right) = 2^M \Phi(2z, 1, M) - \Phi(z, 1, M).$$

Remark 1.2. (cf. [1]) In Theorem 1.1, by putting $M = 2$, we respectively have

$$\frac{1}{1-z} \left\{ \Phi\left(\frac{z}{1-z}, 1, 1\right) + \Phi\left(\frac{z}{1-z}, 1, 2\right) \right\} = 4\Phi(2z, 1, 2) - \Phi(z, 1, 2).$$

2. Main Results

In this section, by using the integral representation of Hurwitz-Lerch zeta function with order 1, we give an integral formula as corollary.

Suppose that $x \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $|\lambda| < 1$. We set

$$\frac{x^2 + 3x + 2}{(x+1)(x+2)} - \frac{2}{(x+1)(x+2)} + \sum_{m=2}^{\infty} \lambda^{m-1} = \frac{1}{1-\lambda} - \frac{2}{(x+1)(x+2)}$$

$$\Updownarrow$$

$$\frac{x(x+3)}{(x+1)(x+2)}+\sum_{m=2}^{\infty}\lambda^{m-1}=\frac{1}{1-\lambda}-\frac{2}{(x+1)(x+2)}$$

$$\Updownarrow$$

$$\frac{x(x+3)\lambda}{(x+1)(x+2)}+\sum_{m=2}^{\infty}\lambda^{m}=\frac{\lambda}{1-\lambda}-\frac{2\lambda}{(x+1)(x+2)}$$

$$\Updownarrow$$

$$x\sum_{m=1}^{\infty}\left\{\frac{x+3m}{(x+m)(x+2m)}\right\}\lambda^{m}+2\sum_{m=2}^{\infty}\left\{\frac{m^{2}}{(x+m)(x+2m)}\right\}\lambda^{m}=\frac{\lambda}{1-\lambda}-\frac{2\lambda}{(x+1)(x+2)}. \tag{2.1}$$

Suppose that

$$\frac{1}{(x+m)(x+2m)}=\frac{A}{x+m}+\frac{B}{x+2m}$$

or

$$1=A(x+2m)+B(x+m).$$

Then, we get $A=1/m$ and $B=-1/m$.Therefore,

$$\frac{1}{(x+m)(x+2m)}=\frac{1}{m}\left\{\frac{1}{x+m}-\frac{1}{x+2m}\right\}. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned} & x\sum_{m=1}^{\infty}\left\{\frac{(x+3m)}{m}\left(\frac{1}{x+m}-\frac{1}{x+2m}\right)\right\}\lambda^{m}+2\sum_{m=2}^{\infty}m\left(\frac{1}{x+m}-\frac{1}{x+2m}\right)\lambda^{m} \\ = & \frac{\lambda}{1-\lambda}-\frac{2\lambda}{(x+1)(x+2)} \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} & x^{2}\sum_{m=1}^{\infty}\left\{\frac{1}{x+m}-\frac{1}{2\left(\frac{x}{2}+m\right)}\right\}\frac{\lambda^{m}}{m}+3x\sum_{m=1}^{\infty}\left\{\frac{1}{x+m}-\frac{1}{2\left(\frac{x}{2}+m\right)}\right\}\lambda^{m} \\ = & \frac{\lambda}{1-\lambda}-\frac{2\lambda}{(x+1)(x+2)}-2\sum_{m=2}^{\infty}m\left\{\frac{1}{x+m}-\frac{1}{2\left(\frac{x}{2}+m\right)}\right\}\lambda^{m} \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} & x^{2}\sum_{m=0}^{\infty}\left\{\frac{1}{x+m+1}-\frac{1}{2\left(\frac{x}{2}+m+1\right)}\right\}\frac{\lambda^{m+1}}{m+1}+3x\sum_{m=0}^{\infty}\left\{\frac{1}{x+m+1}-\frac{1}{2\left(\frac{x}{2}+m+1\right)}\right\}\lambda^{m+1} \\ = & \frac{\lambda}{1-\lambda}-\frac{2\lambda}{(x+1)(x+2)}-2\sum_{m=0}^{\infty}(m+2)\left\{\frac{1}{x+m+2}-\frac{1}{2\left(\frac{x}{2}+m+2\right)}\right\}\lambda^{m+2} \end{aligned}$$

$$\Updownarrow$$

$$
\begin{aligned}
& x^2\left\{\int_0^\lambda\left(\Phi(t,1,x+1)-\frac{\Phi\left(t,1,\frac{x}{2}+1\right)}{2}\right)dt\right\}+3\lambda x\left\{\Phi(\lambda,1,x+1)-\frac{\Phi\left(\lambda,1,\frac{x}{2}+1\right)}{2}\right\}\\
= & \frac{\lambda}{1-\lambda}-\frac{2\lambda}{(x+1)(x+2)}-2\lambda\frac{\partial}{\partial\lambda}\left\{\lambda^2\left(\Phi(\lambda,1,x+2)-\frac{\Phi\left(\lambda,1,\frac{x}{2}+2\right)}{2}\right)\right\}.
\end{aligned}
$$

Then, we obtain the following theorem:

Theorem 2.1. *Let $x \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $|\lambda| < 1$. Then, we have*

$$
\begin{aligned}
& \frac{1}{1-\lambda}-\frac{2}{(x+1)(x+2)}-\frac{x^2}{\lambda}\left\{\int_0^\lambda\left(\Phi(t,1,x+1)-\frac{\Phi\left(t,1,\frac{x}{2}+1\right)}{2}\right)dt\right\}\\
= & 3x\left\{\Phi(\lambda,1,x+1)-\frac{\Phi\left(\lambda,1,\frac{x}{2}+1\right)}{2}\right\}+4\lambda\left\{\Phi(\lambda,1,x+2)-\frac{\Phi\left(\lambda,1,\frac{x}{2}+2\right)}{2}\right\}\\
& +2\lambda^2\frac{\partial}{\partial\lambda}\left\{\Phi(\lambda,1,x+2)-\frac{\Phi\left(\lambda,1,\frac{x}{2}+2\right)}{2}\right\}.
\end{aligned}
$$

For $|z| < 1$ and $w \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the integral representation of the Hurwitz-Lerch zeta function with order 1 is given by

$$
\Phi(\lambda,1,x)=\int_0^1\frac{u^{x-1}}{1-\lambda u}du. \tag{2.3}
$$

Also, by using partial derivative, we get

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}\left\{\lambda^2\Phi(\lambda,1,x)\right\} &= \frac{\partial}{\partial\lambda}\left\{\int_0^1\frac{\lambda^2u^{x-1}}{1-\lambda u}du\right\}\\
&= \int_0^1\frac{(2\lambda-u\lambda^2)u^{x-1}}{(1-\lambda u)^2}du.
\end{aligned} \tag{2.4}
$$

By using (2.3) and (2.4) into Theorem 2.1,we have

$$
\begin{aligned}
& \frac{1}{1-\lambda}-\frac{2}{(x+1)(x+2)}-\frac{x^2}{\lambda}\int_0^\lambda\left\{\int_0^1\frac{u^x}{1-tu}du-\frac{1}{2}\int_0^1\frac{u^{x/2}}{1-tu}du\right\}dt\\
= & 3x\left\{\int_0^1\frac{u^x}{1-\lambda u}du-\frac{1}{2}\int_0^1\frac{u^{x/2}}{1-\lambda u}du\right\}+4\lambda\left\{\int_0^1\frac{u^{x+1}}{1-\lambda u}du-\frac{1}{2}\int_0^1\frac{u^{(x/2)+1}}{1-\lambda u}du\right\}\\
& +2\lambda^2\frac{\partial}{\partial\lambda}\left\{\int_0^1\frac{u^{x+1}}{1-\lambda u}du-\frac{1}{2}\int_0^1\frac{u^{(x/2)+1}}{1-\lambda u}du\right\}
\end{aligned}
$$

$$\Updownarrow$$

$$\begin{aligned}
&\frac{1}{1-\lambda}-\frac{2}{(x+1)(x+2)}-\frac{x^2}{\lambda}\int_0^1\left(u^x-\frac{u^{x/2}}{2}\right)\left\{\int_0^\lambda\frac{dt}{1-tu}\right\}du\\
=\;&3x\int_0^1\left(u^x-\frac{u^{x/2}}{2}\right)\frac{1}{1-\lambda u}du+2\lambda\int_0^1\left(u^{x+1}-\frac{u^{(x/2)+1}}{2}\right)\frac{(2-\lambda u)}{(1-\lambda u)^2}du.
\end{aligned}$$

We note that

$$\begin{aligned}
\int_0^\lambda\frac{dt}{1-tu} &= -\frac{1}{u}\int_0^\lambda\frac{(-u)}{1-tu}dt\\
&= -\frac{\ln(1-\lambda u)}{u}.
\end{aligned}$$

Consequently, we arrive at the following corollary:

Corollary 2.2. *Let $x\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $|\lambda|<1$. Then, we have*

$$\begin{aligned}
&\frac{1}{1-\lambda}-\frac{2}{(x+1)(x+2)}-\frac{x^2}{\lambda}\int_0^1\left(u^x-\frac{u^{x/2}}{2}\right)\ln(1-\lambda u)du\\
=\;&3x\int_0^1\left(u^x-\frac{u^{x/2}}{2}\right)\frac{1}{1-\lambda u}du+2\lambda\int_0^1\left(u^{x+1}-\frac{u^{(x/2)+1}}{2}\right)\frac{(2-\lambda u)}{(1-\lambda u)^2}du.
\end{aligned}$$

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