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The Faber Polynomial Expansion Method for a Subclass of Analytic and Bi-Univalent Functions Associated with the Janowski Functions

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Dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday

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Abstract

In this present investigation, we introduce a new subclass of analytic and bi-univalent functions associated with Janowski functions. Using the Faber polynomial expansions, we determine a general coefficients bounds $|a_n|$, $n \geq 3$ for this newly defined class. Relevant connections of the results presented in this paper with those in a number of other related works on this subject are also pointed out.

Keywords: Analytic functions, Univalent functions, Bi-univalent functions, Faber polynomial expansion

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1. Introduction and Definitions

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions analytic in the open unit disc

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$$

and let $\mathcal{A}$ be the subclass of functions f in $\mathcal{H}(\mathbb{U})$ satisfying the normalization condition

$$f(0) = f'(0) - 1 = 0.$$

Thus, the functions in $\mathcal{A}$ are represented by the Taylor-Maclaurin series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

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Let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}$. Furthermore, let $\mathcal{P}$ denotes the class of analytic functions ϕ normalized by

$$\phi(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad (z \in \mathbb{U}), \tag{1.2}$$

such that

$$\Re(\phi(z)) > 0.$$

Definition 1.1. A function $f \in \mathcal{A}$ is said to be subordinate to a function g written as $f \prec g$, if there exist a Schwarz function w with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad \text{such that} \quad f = g(w).$$

In particular if g is univalent in $\mathbb{U}$, then

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.2. A function p is said to be in the class $\mathcal{P}[A, B]$ $(-1 \leq B < A \leq 1)$ if it is analytic in $\mathbb{U}$ with $p(0) = 1$ and

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \quad (z \in \mathbb{U}).$$

Geometrically a function p belongs to $\mathcal{P}[A, B]$ maps the open unit disc $\mathbb{U}$ onto the disc defined by the domain

$$\Omega[A, B] = \left\{ w : \left| w - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}.$$

The class $\mathcal{P}[A, B]$ connected with the class $\mathcal{P}$ of functions with positive real part by the relation

$$p(z) \in \mathcal{P}, \text{ if and only if } \frac{(A + 1)p(z) - (A - 1)}{(B + 1)p(z) - (B - 1)} \in \mathcal{P}[A, B].$$

The class $\mathcal{P}[A, B]$ was introduced by Janowski [10]. This functions class is further generalized by the many well-known authors (see, for example [11, 15, 23, 19, 24]).

The Koebe one-quarter theorem shows that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk $\{w : |w| < \frac{1}{4}\}$ of radius $\frac{1}{4}$. Every univalent function f has an inverse f^{-1} defined on some disk containing the disk $\{w : |w| < \frac{1}{4}\}$ and satisfying:

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \tag{1.3}$$

A function $f \in \mathcal{S}$ is said to be bi-univalent on $\mathbb{U}$ if f and $g = f^{-1}$ are both univalent on $\mathbb{U}$. We denote the class of all such functions by Σ. In recent years, the pioneering work of Srivastava *et al.* [20] essentially revived the investigation of various subclasses of the analytic and bi-univalent functions class Σ. In fact, in a remarkably large number of sequels to the pioneering work of Srivastava *et al.* [20], several different subclasses of the analytic and bi-univalent functions class Σ were introduced and studied analogously by the many authors (see, for example [4, 5, 6, 17, 12, 27, 26, 28]). However, only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin expansion (1.1) were obtained in these recent papers.

The Faber polynomials introduced by Faber [8] play an important role in various areas of mathematical sciences, especially in Geometric Function Theory of Complex Analysis. Recently, several authors (see, for example [9, 21, 25, 16, 18]) investigated some interesting and useful properties for analytic functions by applying the Faber polynomial expansion method. Motivated by these and other recent work here we define new subclasses of analytic and bi-univalent functions in $\mathbb{U}$ and (by means of the Faber polynomial expansion method) we determine estimates for the general coefficients $|a_n|$ $(n \geq 3)$ in the Taylor-Maclaurin series expansion (1.1) of functions in each of these subclasses.

We now introduce the following subclass of the analytic and bi-univalent functions class Σ.

Definition 1.3. Let $b \in \mathbb{C}\backslash\{0\}$, $\beta \geq 0$, $\lambda \geq 0$ and $-1 \leq B < A \leq 1$. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{R}[b, \beta, \lambda, A, B]$, if and only if

$$1 + \frac{1}{b}\left\{\mathcal{L}(f, \beta, \lambda; z) - 1\right\} \prec \frac{1 + Az}{1 + Bz}, \; z \in \mathbb{U}, \tag{1.4}$$

and

$$1 + \frac{1}{b}\left\{\mathcal{M}(g, \beta, \lambda; w) - 1\right\} \prec \frac{1 + Aw}{1 + Bw}, \; w \in \mathbb{U}. \tag{1.5}$$

where

$$\begin{aligned} \mathcal{L}(f, \beta, \lambda; z) &= \frac{(1-\beta)\, f(z) + \beta\,(zf'(z)) + \lambda z^2 f''(z)}{z}, \\ \mathcal{M}(g, \beta, \lambda; w) &= \frac{(1-\beta)\, g(w) + \beta\left(wg'(w) + \lambda w^2 g''(w)\right)}{w} \end{aligned}$$

and $g(w) = f^{-1}(w)$ defined in (1.3).

2. The Faber Polynomial Expansion Method and Its Applications

In this section, by using the Faber polynomial expansion of a function f of the form (1.1), we observe that the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows;

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} = &\frac{(-n)!}{(-2n+1)!(n-5)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6}\left[a_6 + (-2n+5)a_3 a_4\right] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

Here, and in what follows, such expressions as (for example) $(-n)!$ are to be interpreted symbolically by

$$(-n)! = \Gamma(1-n) = (-n)(-n-1)(-n-2)\ldots\cdot$$

and V_j $(7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, .., a_n$ [3]. In particular, the first three terms of K_{n-1}^{-n} are given below

$$\frac{1}{2} K_1^{-2} = -a_2, \tag{2.1}$$

$$\frac{1}{3} K_2^{-3} = 2a_2^2 - a_3 \tag{2.2}$$

and

$$\frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4). \tag{2.3}$$

In general, an expansion of K_n^p is given by (see, for details, [2]) ,

$$K_n^p = pa_n + \frac{p(p-1)}{2} E_{n-1}^2 + \frac{p!}{(p-3)!3!} E_{n-1}^3 + ... + \frac{p!}{(p-n+1)!(n-1)!} E_{n-1}^{n-1} \quad p \in \mathbb{Z}, \tag{2.4}$$

where $\mathbb{Z} = \{0, \pm1, \pm2, ...\}$ (see [13]).

$$E_n^p = E_n^p(a_2, a_3....).$$

It is clearly seen that

$$E_{n-1}^{m}(a_2,...,a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1}...(a_n)^{\mu_{n-1}}}{\mu_{1!},...,\mu_{n-1}!} \qquad (m \leq n)\,.$$

While $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1,...,\mu_n$ satisfying:

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$

and

$$\mu_1 + 2\mu_2 + ... + (n-1)\mu_{n-1} = n-1.$$

Evidently, (see [1])

$$E_{n-1}^{n-1}(a_2,...,a_n) = a_2^{n-1},$$

or equivalently,

$$E_{n}^{m}(a_{1,}a_2,...,a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1}...(a_n)^{\mu_n}}{\mu_{1!},...,\mu_n!}, \quad m \leq n,$$

again $a_1 = 1$, and taking the sum over all nonnegative integer $\mu_1,...,\mu_n$ satisfying:

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$

and

$$\mu_1 + 2\mu_2 + ... + (n)\mu_n = n.$$

It is clear that

$$E_n^n(a_1,...,a_n) = E_1^n$$

the first and last polynomials are

$$E_n^n = a_1^n \quad \text{and} \quad E_n^1 = a_n.$$

Now before to state and prove our main results we need the following lemma.

Lemma 2.1. *[9] Let the function $w \in \mathcal{A}$ given by*

$$w(z) = \sum_{n=1}^{\infty} w_n z^n$$

be a Schwarz function so that

$$|w(z)| < 1 \quad for \quad |z| < 1.$$

If $\gamma \geq 0$ then

$$\left| w_2 + \gamma w_1^2 \right| \leq 1 + (\gamma - 1)\, |w_1|^2\,.$$

Theorem 2.2. *For $b \in \mathbb{C}\backslash\{0\}$, $\beta \geq 0$, $\lambda \geq 0$,and $-1 \leq B < A \leq 1$. Let $f \in \mathcal{R}[b,\beta,\lambda,A,B]$, if $a_m = 0$, $2 \leq m \leq n-1$, then*

$$|a_n| \leq \frac{|b|(A-B)}{1+(n-1)(\beta+n\lambda)} \qquad (n \geq 3)\,. \tag{2.5}$$

Proof. For the function $f \in \mathcal{R}[b,\beta,\lambda,A,B]$ of the form (1.1), we have

$$1 + \frac{1}{b}\left\{\mathcal{L}(f,\beta,\lambda;z) - 1\right\} = 1 + \sum_{n=2}^{\infty} \frac{1+(n-1)(\beta+n\lambda)}{b} a_n z^{n-1}, \tag{2.6}$$

and for its inverse map $g = f^{-1}$, we have

$$1 + \frac{1}{b}\{\mathcal{M}(g,\beta,\lambda;w) - 1\} = 1 + \sum_{n=2}^{\infty} \frac{1 + (n-1)(\beta + n\lambda)}{b} A_n w^{n-1}, \tag{2.7}$$

where,

$$A_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...a_n).$$

Since, both functions f and its inverse $g = f^{-1}$ are in $\mathcal{R}[b,\beta,\lambda,A,B]$, by the definition of subordination, there exist two Schwarz functions

$$u(z) = \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{U})$$

and

$$v(w) = \sum_{n=1}^{\infty} d_n w^n \quad (w \in \mathbb{U}).$$

So that we have

$$\begin{aligned} 1 + \frac{1}{b}\{\mathcal{L}(f,\beta,\lambda;z) - 1\} &= \frac{1 + A(u(z))}{1 + B(u(z))} \\ &= 1 - \sum_{n=1}^{\infty}(A - B)K_n^{-1}(c_1, c_2, ..., c_n, B)z^n \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} 1 + \frac{1}{b}\{\mathcal{M}(g,\beta,\lambda;w) - 1\} &= \frac{1 + A(v(w))}{1 + B(v(w))} \\ &= 1 - \sum_{n=1}^{\infty}(A - B)K_n^{-1}(d_1, d_2, ..., d_n, B)w^n. \end{aligned} \tag{2.9}$$

In general (see, for example [2, 3]) for any $p \in N$ and $n \geq 2$, an expansion of $K_n^p(k_1, k_2, ..., k_n, B)$,

$$\begin{aligned} K_n^p(k_1, k_2, ..., k_n, B) &= \frac{p!}{(p-n)!n!} k_1^n B^{n-1} + \frac{p!}{(p-n+1)!(n-2)!} k_1^{n-2} k_2 B^{n-2} \\ &+ \frac{p!}{(p-n+2)!(n-3)!} \times k_1^{n-3} k_3 B^{n-3} \\ &+ \frac{p!}{(p-n+3)!(n-4)!} k_1^{n-4} \left[k_4 B^{n-4} + \frac{p-n+3}{2} k_3^2 B \right] \\ &+ \frac{p!}{(p-n+4)!(n-5)!} k_1^{n-5} \left[k_5 B^{n-5} + (p-n+4) k_3 k_4 B \right] + \sum_{j \geq 6} k_1^{n-1} X_j, \end{aligned}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_1, k_2, ..., k_n$.

For the coefficients of the Schwarz functions $u(z)$ and $v(w)$ we have (see, for example [7])

$$|c_n| \leq 1 \quad \text{and} \quad |d_n| \leq 1.$$

Comparing the corresponding coefficients of (2.6) and (2.8), we have

$$\frac{1 + (n-1)(\beta + n\lambda)}{b} a_n = -(A - B)K_{n-1}^{-1}(c_1, c_2, ..., c_{n-1}, B), \tag{2.10}$$

Similarly corresponding coefficients of (2.7) and (2.9), we have

$$\frac{1 + (n-1)(\beta + n\lambda)}{b} A_n = -(A - B)K_n^{-1}(d_1, d_2, ..., d_n, B) \tag{2.11}$$

Note that for $a_m = 0$; $2 \leq m \leq n-1$, we have $A_n = -a_n$

$$\frac{1+(n-1)(\beta+n\lambda)}{b} a_n = -(A-B)c_{n-1}, \tag{2.12}$$

$$\frac{1+(n-1)(\beta+n\lambda)}{b} a_n = (A-B)d_{n-1}. \tag{2.13}$$

Taking the absolute values of (2.12) and (2.13), we have

$$|a_n| \leq \frac{b(A-B)}{1+(n-1)(\beta+n\lambda)} |c_{n-1}| = \frac{b(A-B)}{1+(n-1)(\beta+n\lambda)} |d_{n-1}|,$$

$$|a_n| \leq \frac{b(A-B)}{1+(n-1)(\beta+n\lambda)}, \qquad n \geq 3.$$

which completes the proof of Theorem 2.2. □

If we set

$$\beta = b = 1 = -B = A + 2\alpha \quad (0 \leq \alpha < 1)$$

in Theorem 2.2, we obtain the following known result.

Corollary 2.3. *[22] Let* $f \in \mathcal{R}[1, 1, \lambda, 1-2\alpha, -1]$, *if* $a_m = 0$, $2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{2(1-\alpha)}{n\left[1+\lambda(n-1)\right]}, \qquad n \geq 3.$$

Theorem 2.4. *For* $b \in \mathbb{C}\backslash\{0\}$, $\beta \geq 0$, $\lambda \geq 0$ *and* $-1 \leq B < A \leq 1$. *Let* $f \in \mathcal{R}[b, \beta, \lambda, A, B]$ *then*

$$|a_2| \leq \begin{cases} \frac{b(A-B)}{\sqrt{b(A-B)\{1+2(\beta+3\lambda)\}+(1+B)\{1+(\beta+2\lambda\}^2}}, & \text{if} \quad 0 \leq B < A, \\ \frac{b(A-B)}{1+(\beta+2\lambda)}, & \text{otherwise}. \end{cases}$$

and

$$\left|a_3 - 2a_2^2\right| \leq \frac{b^2(A-B)^2-(B+1)\{1+\beta+2\lambda\}^2}{b(A-B)\left[1+2(\beta+3\lambda)\right]} |a_2|^2, \qquad \text{if} \quad A \leq 0,$$

$$\left|a_3 - a_2^2\right| \leq \frac{b(A-B)}{(1+2(\beta+3\lambda))}, \qquad \text{if} \ A > 0.$$

Proof. Replacing n by 2 and 3 in (2.10) and (2.11), respectively, we have

$$\frac{1+(\beta+2\lambda)}{b} a_2 = (A-B)c_1, \tag{2.14}$$

$$\frac{1+2(\beta+3\lambda)}{b} a_3 = (A-B)(Bc_1^2 - c_2), \tag{2.15}$$

$$-\frac{1+(\beta+2\lambda)}{b} a_2 = (A-B)d_1, \tag{2.16}$$

$$\frac{1+2(\beta+3\lambda)}{b} \left\{2a_2^2 - a_3\right\} = (A-B)(Bd_1^2 - d_2). \tag{2.17}$$

From (2.14) and (2.16) we have

$$|a_2| \leq \frac{b(A-B)}{1+(\beta+2\lambda)} |c_1| = \frac{b(A-B)}{1+(\beta+2\lambda)} |d_1|,$$

$$\leq \frac{b(A-B)}{1+(\beta+2\lambda)}. \tag{2.18}$$

Adding (2.15) and (2.17) we have

$$\frac{2\left(1+2(\beta+3\lambda)\right)}{b}a_2^2=(A-B)\left\{(Bc_1^2-c_2)+(Bd_1^2-d_2)\right\}. \tag{2.19}$$

Taking absolute values of (2.19), we obtain

$$\frac{2\left(1+2(\beta+3\lambda)\right)}{b}\left|a_2\right|^2=(A-B)\left\{\left|c_2+(-B)c_1^2\right|+\left|d_2+(-B)\,d_1^2\right|\right\}$$

If $B\leq 0$, then by Lemma 2.1, we have

$$\frac{2\left(1+2(\beta+3\lambda)\right)}{b}\left|a_2\right|^2\leq(A-B)\left\{1+(-B-1)\left|c_1\right|^2+1+(-B-1)\left|d_1\right|^2\right\}. \tag{2.20}$$

By using

$$\left(\frac{1+(\beta+2\lambda)}{b(A-B)}\right)^2\left|a_2\right|^2=\left|c_1\right|^2=\left|d_1\right|^2,$$

we can get that

$$\left|a_2\right|\leq\frac{b(A-B)}{\sqrt{b(A-B)\left\{1+2(\beta+3\lambda)\right\}+(1+B)\{1+(\beta+2\lambda\}^2}}.$$

Obviously, for $A>0$ we have

$$\frac{b(A-B)}{\sqrt{b(A-B)\left\{1+2(\beta+3\lambda)\right\}+(1+B)\{1+(\beta+2\lambda\}^2}}<\frac{b(A-B)}{1+(\beta+2\lambda)}.$$

Rewriting equation (2.17), we have

$$\frac{1+2(\beta+3\lambda)}{b}\left\{a_3-2a_2^2\right\}=(A-B)\left\{d_2-Bd_1^2\right\}.$$

Taking the absolute values of the above equation, we have

$$\left|\frac{1+2(\beta+3\lambda)}{b}\left\{a_3-2a_2^2\right\}\right|=(A-B)\left|\left\{d_2-Bd_1^2\right\}\right|.$$

we can get that If $A\leq 0$, then by Lemma 2.1 and

$$a_2^2=\left(\frac{b(A-B)}{1+(\beta+2\lambda)}\right)^2d_1^2,$$

we have

$$\left|a_3-2a_2^2\right|\leq\frac{b^2\left(A-B\right)^2-(B+1)\left\{1+\beta+2\lambda\right\}^2}{b\left(A-B\right)\left[1+2\left(\beta+3\lambda\right)\right]}\left|a_2\right|^2.$$

By using

$$\left(\frac{1+(\beta+2\lambda)}{b(A-B)}\right)^2\left|a_2\right|^2=\left|d_1\right|^2,$$

for $A>0$, subtracting (2.17) from (2.15), we have

$$\frac{2\left(1+2(\beta+3\lambda)\right)}{b}\left(a_3-a_2^2\right)=(A-B)\left[B(d_1^2-c_1^2)+(c_2-d_2)\right].$$

Using the fact that $c_1^2=d_1^2$ and taking the absolute values of both sides of the above equation, we obtain the desired inequality

$$\left|a_3-a_2^2\right|\leq\frac{b(A-B)}{(1+2(\beta+3\lambda))}.$$

This completes the proof. □

Acknowledgments

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