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Euclidean Degree Energy Graphs

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Abstract

In this paper we introduce new energy of graph that is Euclidean degree energy. We obtain characteristic polynomial of the Euclidean degree of standard graphs and graphs obtained by some graph operations and also we characterize Euclidean hyperenergetic, nonhyperenergetic and borderenergetic graphs.

Keywords: Euclidean degree matrix, Euclidean degree polynomial and energy, Hyperenergetic graphs

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1. Introduction

A graph G is a finite nonempty set of points called vertices, together with a set of unordered pairs of distinct vertices called edges. Let $V(G)$ be the vertex set and $E(G)$ be an edge set of G. The set of edges may be empty. The degree of a vertex u, $d_u(G)$, is the number of edges incident on u. A graph is a regular graph if all the vertices of the graph have equal degrees. A graph is considered a complete graph if each pair of vertices is joined by an edge. For more basic terminologies and notations we referred [12]. Let $A(G) = (a_{ij})$ be an adjacency matrix of order n of a graph G. The characteristic polynomial of a graph G is denoted by $Ch(A(G), \lambda) =| \lambda I - A(G) |$, where λ is an eigenvalue of a graph G. Hence, by [10], the energy of G is defined as $\mathcal{E}(G) = \sum_{i=1}^{n} | \lambda_i |$.
Bapat and Pati [2] have proved that if the energy of a graph is rational then it must be an even integer and Pirzada and Gutman [16] showed that the energy of a graph is never the square root of an odd integer. Initially, the concept of energy in a graph arose from Huckel theory in which the π-electron energy of a conjugated carbon molecule was computed, which coincides with the energy of a graph. The Euclidean degree square sum matrix of a graph G is denoted by $EDE(G) = (s_{ij})$ and whose elements are defined as

$$s_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2} & \text{if } v_i \sim v_j \\ 0 & \text{if otherwise} \end{cases} .$$

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2. Some basic properties of largest Euclidean degree eigenvalue

Let us define number p as

$$p = \sum_{i<j}(d_i^2 + d_j^2)$$

Proposition 2.1. *The first three coefficient of the polynomial $Ch(EDE(G,\lambda))$ are as follows*

(i) $a_0 = 1$

(ii) $a_1 = 0$

(iii) $a_2 = -p$

Proof. (i) By the definition of characteristic polynomial we get, $a_0 = 1$

(ii) Sum of all principal diagonal entries of Euclidean degree matrix is equal to the trace of $EDE(G)$. Thus,

$$a_1 = tr(EDE(G)) = 0$$

(iii) We have ,

$$(-1)^2 a_2 = \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

$$= \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ji}a_{ij})$$

$$= -p$$

□

Proposition 2.2. *If λ_1, λ_2 ,..., λ_n are the Euclidean degree eigenvalues of EDE(G) then,*

$$\sum_{i=1}^{n} \lambda_i^2 = 2p$$

Proof.

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji}$$

$$= 2\sum_{i<j}(a_{ij})^2 + \sum_{i=1}^{n}(a_{ii})^2$$

$$= \sum_{i<j}(a_{ij})^2$$

$$= 2p$$

□

Theorem 2.3 ([15]). *Let a_i and b_i be non-negative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \frac{n^2}{4}(M_1M_2 - m_1m_2)^2 \tag{2.1}$$

where, $M_1 = max(a_i)$, $M_2 = max(b_i), m_1 = min(a_i)$, $m_2 = min(b_i)$ where $i = 1, 2, ..., n$

Theorem 2.4 ([**?**]). *Let a_i and b_i be non-negative real numbers, then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b) \tag{2.2}$$

where a, b, A and B are real constants such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for each i, $1 \leq i \leq n$. Further, $\alpha(n) = n\lfloor \frac{n}{2} \rfloor (1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Theorem 2.5 ([8]). *Let a_i and b_i be non-negative real numbers, then*

$$\sum_{i=1}^{n} b_i^2 + C_1C_2 \sum_{i=1}^{n} a_i^2 \leq (C_1 + C_2) \sum_{i=1}^{n} a_i b_i \tag{2.3}$$

where C_1 and C_2 are real constants such that $C_1 a_i \leq b_i \leq C_2 a_i$ for each i, $1 \leq i \leq n$.

Theorem 2.6. *Let G be an r-regular graph of order n. Then G has only one positive Euclidean degree eigenvalue $\lambda = \sqrt{2}r(n-1)$.*

Proof. Let G be a connected r-regular graph of order n and $\{v_1, v_2, ..., v_n\}$ be the vertex set of G. Let $d_i = r$ be the degree of v_i, $i = 1, 2, ...n$. Then the characteristic polynomial of $EDE(G)$

$$Ch[EDE(G), \lambda] = (\lambda - \sqrt{2}r(n-1))(\lambda + \sqrt{2}r)^{n-1} \tag{2.4}$$

Therefore, the eigenvalues are $\sqrt{2}r(n-1)$ and $-\sqrt{2}r$ which repeats $(n-1)$ times.

□

Theorem 2.7. *Let G be any graph of order n and λ_1 be the largest Euclidean degree eigenvalue. Then*

$$\lambda_1 \leq \sqrt{\frac{2p(n-1)}{n}}$$

Proof. By the Cauchy-Schwartz inequality [[**?**]] we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

where a_i and b_i are non-negative real numbers.
now, substituting $a_i = 1$ and $b_i = \lambda_i$,we have

$$\left(\sum_{i=2}^{n} \lambda_i^2\right)^2 \leq (n-1) \sum_{i=2}^{n} \lambda_i^2$$

By using propositions 2.1 and 2.2 in above inequality

$$(-\lambda_1)^2 \leq (n-1)(2p - \lambda_1^2)$$

Hence,

$$\lambda_1 \leq \sqrt{\frac{2p(n-1)}{n}}$$

Remark 2.8. If G is an regular graph then

$$\lambda_1 = \sqrt{\frac{2p(n-1)}{n}}$$

□

Remark 2.9. Let G be an r-regular graph of order n , then $EDE(G) = r^2J - r^2I$. Where J is the the matrix of order n whose all entries are equal to one and I is an identity matrix of order n.
The characteristic polynomial is given by

$$Ch[EDE(G), \lambda] = (\lambda - \sqrt{2}r(n-1))(\lambda + \sqrt{2}r)^{n-1}$$

Hence ,

$$\mathcal{E}[EDE(G)] = 2\sqrt{2}r(n-1) \tag{2.5}$$

Remark 2.10. If G is an r-regular graph , its complement $\overline{G}$ is $(n-1-r)$ regular graph then we have,

$$Ch[EDE(\overline{G}), \lambda] = (\lambda - \sqrt{2}(n-1)(n-1-r))(\lambda + \sqrt{2}(n-1-r))^{n-1}$$

Thus ,

$$\mathcal{E}[EDE(\overline{G})] = 2\sqrt{2}(n-1-r)(n-1) \tag{2.6}$$

Theorem 2.11. *Let G be an graph of order n and size m. Then*

$$\mathcal{E}[EDE(G)] \geq \sqrt{2np - \frac{n^2}{4}(|\lambda_1| - |\lambda_2|)^2}$$

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of $EDE(G)$. Substituting $a_i = 1$ and $b_i =|\lambda_i|$ in the equation (1) We get

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^2 \leq \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2$$

$$2pn - (\mathcal{E}[EDE(G)])^2 \leq \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2$$

$$\mathcal{E}[EDE(G)] \geq \sqrt{2np - \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2}$$

□

Corollary 2.12. *If G is an r-regular graph of order n, then*

$$\mathcal{E}[EDE(G)] \geq nr^2\sqrt{8(n-1) - n^2}$$

Theorem 2.13. *Let G be an graph of order n, then*

$$\sqrt{2p} \leq \mathcal{E}[EDE(G)] \leq \sqrt{2np}$$

Proof. By the Cauchy-Schwartz inequality [[**?**]] we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

where a_i and b_i are non-negative real numbers.
Now, substituting $a_i = 1$ and $b_i = \lambda_i$ we have

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\lambda_i|^2$$

$$(\mathcal{E}[EDE(G)])^2 \leq 2pn$$

Thus,

$$\mathcal{E}[EDE(G)] \leq \sqrt{2pn}$$

and

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \left(\sum_{i=1}^{n} |\lambda_i|\right)^2$$

$$2p \leq (\mathcal{E}[EDE(G)])^2$$

which implies

$$\mathcal{E}[EDE(G)] \geq \sqrt{2p}$$

□

Theorem 2.14. *Let G be a graph of order n and* Δ *be the absolute value of the determinant of* $EDE(G)$*. Then*

$$\sqrt{2p + n(n-1)\Delta^{\frac{2}{n}}} \leq \mathcal{E}[EDE(G)] \leq \sqrt{2np}$$

Proof.

$$(\mathcal{E}[EDE(G)])^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2$$

$$= \sum_{i=1}^{n} \lambda_i^2 + 2\sum_{i<j} |\lambda_i||\lambda_j|$$

$$= 2p + 2\sum_{i<j} |\lambda_i||\lambda_j|$$

$$(\mathcal{E}[EDE(G)])^2 = 2p + \sum_{i\neq j} |\lambda_i||\lambda_j| \tag{2.7}$$

Since we know for non-negative numbers , the arithmetic mean is always greater than or equal to the geometric mean

$$\frac{1}{n(n-1)}\sum_{i\neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{i=1} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= \prod_{i\neq j} |\lambda_i|^{\frac{2}{n}}$$

$$= \Delta^{\frac{2}{n}}$$

Therefore,

$$\sum_{i\neq j} |\lambda_i||\lambda_j| \geq n(n-1)\Delta^{\frac{2}{n}}$$

from equation (7) we have,

$$\mathcal{E}[EDE(G)] \geq \sqrt{2p + n(n-1)\Delta^{\frac{2}{n}}}$$

Consider a non-negative quantity

$$Y = \sum_{i=1}^{n}\sum_{j=1}^{n}(|\lambda_i| - |\lambda_j|)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(|\lambda_i|^2 + |\lambda_j|^2 - 2|\lambda_i||\lambda_j|)$$

$$Y = n\sum_{i=1}^{n} |\lambda_i|^2 + n\sum_{j=1}^{n} |\lambda_j|^2 - 2\sum_{i=1}^{n} |\lambda_i| \sum_{j=1}^{n} |\lambda_j|$$

$$Y = 4np - 2(\mathcal{E}[EDE(G)])^2$$

since

$$Y \geq 0$$

$$4np - 2(\mathcal{E}[EDE(G)])^2 \geq 0$$

$$\mathcal{E}[EDE(G)] \leq \sqrt{2np}$$

□

Corollary 2.15. *If G is an r-regular graph of order n, then*

$$\mathcal{E}[EDE(G)] \leq 2nr^2\sqrt{n-1}$$

Theorem 2.16. *Let G be a graph of order n and size m. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be a non-increasing arrangement Euclidean degree eigenvalues. Then*

$$\mathcal{E}[EDE(G)] \geq \sqrt{2np - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ are the Euclidean degree eigenvalues of G. Substituting $a_i = |\lambda_i| = b_i$ and $a = |\lambda_n| = b$, $A = |\lambda_1| = B$ in the equation (2)

$$\left| n\sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \right| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$$

Since $\mathcal{E}[EDE(G)] = \sum_{i=1}^{n} |\lambda_i|$ and $\sum_{i=1}^{n} |\lambda_i|^2 = 2p$ we get the required result. □

Theorem 2.17. *Let G be a graph of order n and size m. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be a non-increasing arrangement of Euclidean degree eigenvalues. Then*

$$\mathcal{E}[EDE(G)] \geq \frac{|\lambda_1||\lambda_n| n + 2p}{|\lambda_1| + |\lambda_n|}$$

where $|\lambda_1|$ and $|\lambda_2|$ are maximum and minimum of the absolute value of $\lambda_i's$

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be Euclidean degree eigenvalues of G. Substituting $a_i = 1$ and $b_i = |\lambda_i|$, $C_1 = |\lambda_n|$, $C_2 = |\lambda_1|$ in the equation (3)

$$\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} 1^2 \leq (|\lambda_1| + |\lambda_n|)\left(\sum_{i=1}^{n} |\lambda_i|\right)$$

Since $\mathcal{E}[EDE(G)] = \sum_{i=1}^{n} |\lambda_i|$ and $\sum_{i=1}^{n} |\lambda_i|^2 = 2p$ we get the required result. □

Definition 2.18. [12] The line graph $L(G)$ of a graph G is a graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G are adjacent.

The k^{th} iterated line graph [4, 5, 12] of G is defined as $L^k(G) = L(L^{k-1}(G))$, $k = 1, 2, 3..$ where $L^0(G) \cong G$ and $L^1(G) \cong L(G)$

Remark 2.19 ([4, 5]). The line graph $L(G)$ of an r-regular graph of G of order n is an $r_1 = (2r-2)$-regular graph of order $n_1 = \frac{nr}{2}$. Thus, $L^k(G)$ is an r_k-regular graph of order n_k is given by

$$n_k = \frac{n}{2^k}\prod_{i=1}^{k-1}(2^i r - 2^{i+1} + 2) \quad and \quad r_k = 2^k r - 2^{k+1} + 2$$

Theorem 2.20. *Let G be an r-regular graph of order n and let $L^k(G)$ be the r_k-regular graph of order n_k then Euclidean degree energy of $L^k(G)$*

$$\mathcal{E}[EDE(L^k(G))] = 2\sqrt{2}r_k(n-1) \quad where, \quad r_k = 2^k r - 2^{k+1} + 2$$

Proof. The Euclidean degree characteristic polynomial of $L^k(G)$ with vertex set n_k (see remarks 2.9 and 2.18) is given by

$$Ch[EDE(L^k(G)), \lambda] = [\lambda - \sqrt{2}(2^k r - 2^{k+1} + 2)(n_k - 1)][\lambda + \sqrt{2}(2^k r - 2^{k+1} + 2)]^{n_k - 1}$$

Thus,

$$\mathcal{E}[EDE(L^k(G))] = 2\sqrt{2}r_k(n_k - 1) \quad where, \quad r_k = 2^k r - 2^{k+1} + 2$$

□

Lemma 2.21 ([18]). *If a,b,c and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

has the following characteristic equation,

$$= (\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}[(\lambda - (n_1 - 1)a][\lambda - (n_2 - 1)b] - n_1 n_2 cd)]$$

Definition 2.22 ([12]). The subdivision graph $S(G)$ of a graph G is a graph with vertex set $V(G) \cup E(G)$ and is obtained by inserting a new vertex of degree 2 into each edge of G.

Definition 2.23 ([19]). The semitotal line graph $T_1(G)$ of a graph G is a graph with vertex set $V(G) \cup E(G)$ where two vertices of $T_1(G)$ are adjacent if and only if they correspond to two adjacent edges of G or one is a vertex of G and another is an edge G incident with it in G.

Definition 2.24 ([19]). The semitotal point graph $T_2(G)$ of a graph G is a graph with vertex set $V(G) \cup E(G)$ where two vertices of $T_2(G)$ are adjacent if and only if they correspond to two adjacent vertices of G or one is a vertex of G and another is an edge G incident with it in G.

Definition 2.25 ([12]). The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are either adjacent or incident.

Definition 2.26 ([18]). The graph G^{+k} is a graph obtained from the graph G by attaching k pendant edges to each vertex of G. If G is a graph of order n and size m, then G^{+k} is graph of order $n + nk$ and size $m + nk$.

Definition 2.27 ([12]). The union of the graphs G_1 and G_2 is a graph $G_1 \cup G_2$ whose vertex set is $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and the edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 2.28 ([12]). The join $G_1 + G_2$ of two graphs G_1 and G_2 is the graph obtained from G_1 and G_2 by joining every vertex of G_1 to all vertices of G_2.

Definition 2.29 ([12]). The product $G \times H$ of two graphs G and H is defined as follows
Consider any two points $u = (u_1, u_2)$ and $u = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G \times H$ whenever ($u_1 = v_1$ and u_2 adjacent v_2) or ($u_2 = v_2$ and u_1 adjacent v_1).

Definition 2.30 ([12]). The composition $G[H]$ of two graphs G and H is defined as follows: Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G[H]$ whenever [u_1 adj v_1] or [$u_1 = v_1$ and u_2 adjacent v_2].

Definition 2.31 ([12]). The corona $G \circ H$ of graphs G and H is a graph obtained from G and H by taking one copy of G and $| V(G) |$ copies of H and then joining by an edge each vertex of the i^{th} copy of H is named (H, i) with the i^{th} vertex of G.

Definition 2.32 ([**?**]). The jump graph $J(G)$ of a graph G is defined as a graph with vertex set as $E(G)$ where the two vertices of $J(G)$ are adjacent if and only if they correspond to two nonadjacent edges of G.

3. Characteristic polynomials of different graph structures

Theorem 3.1. *Let G be an r- regular graph of order n and size m. Then,*

$$Ch[EDE(S(G))] = (\lambda + \sqrt{2}r)^{n-1}(\lambda + 2\sqrt{2})^{\frac{nr}{2}-1}[\lambda^2 - (2\sqrt{2}(\frac{nr}{2} - 1) + \sqrt{2}r(n-1))\lambda$$
$$+4r(n-1)(\frac{nr}{2} - 1) - \frac{n^2r}{2}(r^2+4)]$$

Proof. The subdivision graph of an r-regular graph has two types of vertices. The n vertices with degree r and $\frac{nr}{2}$ vertices with degree 2. Hence

$$EDE[S(G)] = \begin{bmatrix} \sqrt{2}r(J_n - I_n) & \sqrt{(r^2+4)}J_{n\times\frac{nr}{2}} \\ \sqrt{(r^2+4)}J_{\frac{nr}{2}\times n} & 2\sqrt{2}(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}.$$

$$Ch[EDE(S(G))] =| \lambda I - EDE(S(G)) |$$
$$= \begin{vmatrix} (\lambda + \sqrt{2}r)I_n - \sqrt{2}rJ_n & -\sqrt{(r^2+4)}J_{n\times\frac{nr}{2}} \\ -\sqrt{(r^2+4)}J_{\frac{nr}{2}\times n} & (\lambda + 2\sqrt{2})I_{\frac{nr}{2}} - 2\sqrt{2}J_{\frac{nr}{2}} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.2. *Let G be an r- regular graph of order n and size m. Then,*

$$Ch[EDE(T_2(G))] = (\lambda + 2\sqrt{2}r)^{n-1}(\lambda + 2\sqrt{2})^{\frac{nr}{2}-1}[\lambda^2 - 2\sqrt{2}((\frac{nr}{2} - 1) + r(n-1))\lambda$$
$$+8r(n-1)(\frac{nr}{2} - 1) - 2n^2r(r^2+1))]$$

Proof. The semitotal point graph of a r-regular graph has two types of vertices. The n vertices with degree $2r$ and $\frac{nr}{2}$ vertices with degree 2. Hence

$$EDE(T_2) = \begin{bmatrix} 2\sqrt{2}r(J_n - I_n) & 2\sqrt{r^2+1}J_{n\times\frac{nr}{2}} \\ 2\sqrt{r^2+1}J_{\frac{nr}{2}\times n} & 2\sqrt{2}(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}.$$

$$Ch[EDE(T_2)] =| \lambda I - EDE(T_2(G)) |$$
$$\begin{vmatrix} (\lambda + 2\sqrt{2}r)I_n - 2\sqrt{2}rJ_n & -2\sqrt{r^2+1}J_{n\times\frac{nr}{2}} \\ -2\sqrt{r^2+1}J_{\frac{nr}{2}\times n} & (\lambda + 2\sqrt{2})I_{\frac{nr}{2}} - 2\sqrt{2}J_{\frac{nr}{2}} \end{vmatrix}$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.3. *Let G be an r- regular graph of order n and size m. Then,*

$$Ch[EDE(T_1)] = (\lambda + \sqrt{2}r)^{n-1}(\lambda + 2\sqrt{2}r)^{\frac{nr}{2}-1}[\lambda^2 - r\sqrt{2}(2(\frac{nr}{2} - 1) + (n-1))\lambda + 4r^2(n-1)(\frac{nr}{2} - 1) - \frac{5n^2r^3}{2}]$$

Proof. The semitotal line graph of an r-regular graph has two types of vertices. The n vertices with degree r and $\frac{nr}{2}$ vertices with degree $2r$. Hence

$$EDE(T_1) = \begin{bmatrix} \sqrt{2}r(J_n - I_n) & r\sqrt{5}J_{n\times\frac{nr}{2}} \\ r\sqrt{5}J_{\frac{nr}{2}\times n} & 2\sqrt{2}r(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}.$$

$$Ch[EDE(T_1)] =| \lambda I - EDE(T_1(G)) |$$
$$= \begin{vmatrix} (\lambda + \sqrt{2}r)I_n - \sqrt{2}rJ_n & -r\sqrt{5}J_{n\times\frac{nr}{2}} \\ -r\sqrt{5}J_{\frac{nr}{2}\times n} & (\lambda + 2\sqrt{2}r)I_{\frac{nr}{2}} - 2\sqrt{2}rJ_{\frac{nr}{2}} \end{vmatrix}$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.4. *Let G be an r- regular graph of order n and size m. Then,*

$$Ch[EDE(T(G))] = (\lambda - 2\sqrt{2}r(n + \frac{nr}{2} - 1))(\lambda + 2\sqrt{2}r)^{n+\frac{nr}{2}-1}$$

Proof. The total graph $T(G)$ of an r-regular graph G is a regular graph of degree $2r$ with $n + \frac{nr}{2}$ vertices. Hence the result follows from equation (4).

□

Theorem 3.5. *Let G be an r- regular graph of order n and size m. Then,*

$$Ch[EDE(G^{+k})] = (\lambda + \sqrt{2}(r+k))^{n-1}(\lambda + \sqrt{2})^{nk-1}[\lambda^2 - (\sqrt{2}(nk-1) + \sqrt{2}(r+k)(n-1))\lambda$$
$$+2(r+k)(n-1)(nk-1) - n^2k(1 + (r+k)^2)]$$

Proof. The graph G^{+k} of an r-regular graph of degree $n + nk$ has two types of vertices, with n vertices having degree $r + k$ and nk vertices having degree 1. Hence

$$EDE(G^{+k}) = \begin{bmatrix} \sqrt{2}(r+k)(J_n - I_n) & \sqrt{(r+k)^2+1}J_{n\times nk} \\ \sqrt{(r+k)^2+1}J_{nk\times n} & \sqrt{2}(J_{nk} - I_{nk}) \end{bmatrix}.$$

$$Ch[EDE(G^{+k}))] = \mid \lambda I - EDE(G^{+k}) \mid$$
$$= \begin{vmatrix} (\lambda + \sqrt{2}(r+k))I_n - \sqrt{2}(r+k)J_n & -\sqrt{(r+k)^2+1}J_{n\times nk} \\ -\sqrt{(r+k)^2+1}J_{nk\times n} & (\lambda + \sqrt{2})I_{nk} - \sqrt{2}J_{nk} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.6. *Let G be an r- regular graph of order n and size m. Then,*

$$Ch[EDE(G \cup H)] = Ch(EDE(G))Ch(EDE(H)) - (\lambda + \sqrt{2}r_1)^{n_1-1}(\lambda + \sqrt{2}r_2)^{n_2-1}n_1n_2(r_1^2 + r_2^2)$$

Proof. The graph $G \cup H$ of order $n_1 + n_2$ has two types of vertices, with n_1 vertices having degree r_1 and the remaining n_2 vertices having degree r_2. Hence

$$EDE(G \cup H) = \begin{vmatrix} EDE(G) & \sqrt{(r_1^2 + r_2^2)}J_{n_1\times n_2} \\ \sqrt{(r_1^2 + r_2^2)}J_{n_2\times n_1} & EDE(H) \end{vmatrix}.$$

$$= \begin{vmatrix} \sqrt{2}r_1(J_{n_1} - I_{n_1}) & \sqrt{(r_1^2 + r_2^2)}J_{n_1\times n_2} \\ \sqrt{(r_1^2 + r_2^2)}J_{n_2\times n_1} & \sqrt{2}r_2(J_{n_2} - I_{n_2}) \end{vmatrix}.$$

$$Ch[EDE(G \cup H)] = \mid \lambda I - EDE(G \cup H) \mid$$
$$= \begin{vmatrix} (\lambda + \sqrt{2}r_1)I_{n_1} - \sqrt{2}r_1J_{n_1} & -\sqrt{(r_1^2 + r_2^2)}J_{n_1\times n_2} \\ -\sqrt{(r_1^2 + r_2)^2}J_{n_2\times n_1} & (\lambda + \sqrt{2}r_2)I_{n_2} - \sqrt{2}r_2J_{n_2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get

$$Ch[EDE(G \cup H)] = (\lambda + \sqrt{2}r_1)^{n_1-1}(\lambda + \sqrt{2}r_2)^{n_2-1}[(\lambda - (n_1-1)\sqrt{2}r_1)(\lambda - (n_2-1)\sqrt{2}r_2) - n_1n_2(r_1^2 + r_2^2)]$$

as G and H are regular graphs of order n_1 and n_2 and degree r_1 and r_2 respectively ,by equation (4) we have

$$Ch[EDE(G)] = (\lambda - \sqrt{2}r_1(n_1 - 1))(\lambda + \sqrt{2}r_1)^{n_1-1}$$

and

$$Ch[EDE(H)] = (\lambda - \sqrt{2}r_2(n_2 - 1))(\lambda + \sqrt{2}r_2)^{n_2-1}$$

Hence the result follows. □

Theorem 3.7. *Let G be an r- regular graph of order n and size m. Then,*

$$Ch[EDE(G+H)] = (\lambda + \sqrt{2}R_1)^{n_1-1}(\lambda + \sqrt{2}R_2)^{n_2-1}[\lambda^2 - (\sqrt{2}R_2(n_2-1) + \sqrt{2}R_1(n_1-1))\lambda$$
$$+2R_1R_2(n_1-1)(n_2-1) - n_1n_2(R_1^2+R_2^2)]$$

Proof. If G is an r_1 - regular graph of order n_1 and H is an r_2-regular graph of order n_2 then $G+H$ is a graph of order n_1+n_2 has two types of vertices, the n_1 vertices with degree $R_1 = r_1 + n_2$ and n_2 vertices with degree $R_2 = r_2 + n_1$. Hence

$$EDE(G+H) = \begin{vmatrix} \sqrt{2}R_1(J_{n_1} - I_{n_1}) & \sqrt{(R_1^2+R_2^2)}J_{n_1\times n_2} \\ \sqrt{(R_1^2+R_2^2)}J_{n_2\times n_1} & \sqrt{2}R_2(J_{n_2} - I_{n_2}) \end{vmatrix}.$$

$$Ch[EDE(G+H)] = \mid \lambda I - EDE(G+H) \mid$$

$$= \begin{vmatrix} (\lambda + \sqrt{2}R_1)I_{n_1} - \sqrt{2}R_1J_{n_1} & -\sqrt{(R_1^2+R_2^2)}J_{n_1\times n_2} \\ -\sqrt{(R_1^2+R_2^2)}J_{n_2\times n_1} & (\lambda + \sqrt{2}R_2)I_{n_2} - \sqrt{2}R_2J_{n_2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.8. *Let G be an r_1- regular graph of order n_1 and H be r_2 regular graph of order n_2. Then,*

$$Ch[EDE(G\times H)] = (\lambda - \sqrt{2}(r_1+r_2)(n_1n_2-1))(\lambda + \sqrt{2}(r_1+r_2))^{n_1n_2-1}$$

Proof. Let G be an r_1- regular graph of order n_1 and H be r_2 regular graph of order n_2. Then $G\times H$ is an (r_1+r_2)-regular graph with n_1n_2 vertices. Hence the result follows from equation (4) . □

Theorem 3.9. *Let G be an r_1- regular graph of order n_1 and H be r_2 regular graph of order n_2. Then,*

$$Ch[EDE(G[H])] = (\lambda + \sqrt{2}(n_2r_1+r_2))^{n_1n_2-1}(\lambda - \sqrt{2}(n_2r_1+r_2)(n_1n_2-1))$$

Proof. Let G be an r_1- regular graph of order n_1 and H be r_2 regular graph of order n_2. Then $G[H]$ is an $(n_2r_1+r_2)$-regular graph with n_1n_2 vertices. Hence the result follows from equation (4) . □

Theorem 3.10. *Let G be an r- regular graph of order n and size m. Then,*

$$Ch[EDE(G\circ H)] = (\lambda + \sqrt{2}R_1)^{n_1-1}(\lambda + \sqrt{2}R_2)^{n_2-1}[\lambda^2 - (\sqrt{2}R_2(n_1n_2-1) + \sqrt{2}R_1(n_1-1))\lambda$$
$$+2R_1R_2(n_1-1)(n_1n_2-1) - n_1^2n_2(R_1^2+R_2^2)]$$

Proof. If G is an r_1 - regular graph of order n_1 and H is an r_2-regular graph of order n_2 then $G\circ H$ is a graph of order $n_1+n_1n_2$ has two types of vertices, the n_1 vertices with degree $R_1 = r_1+n_2$ and remaining n_1n_2 vertices with degree $R_2 = r_2+1$. Hence

$$EDE(G\circ H) = \begin{bmatrix} \sqrt{2}R_1(J_{n_1} - I_{n_1}) & \sqrt{(R_1^2+R_2^2)}J_{n_1\times n_1n_2} \\ \sqrt{(R_1^2+R_2^2)}J_{n_1n_2\times n_1} & \sqrt{2}R_2(J_{n_1n_2} - I_{n_1n_2}) \end{bmatrix}.$$

$$Ch[EDE(G\circ H)] = \mid \lambda I - EDE(G\circ H) \mid$$

$$= \begin{vmatrix} (\lambda + \sqrt{2}R_1)I_{n_1} - \sqrt{2}R_1J_{n_1} & -\sqrt{(R_1^2+R_2^2)}J_{n_1\times n_1n_2} \\ -\sqrt{(R_1^2+R_2^2)}J_{n_1n_2\times n_1} & (\lambda + \sqrt{2}R_2)I_{n_1n_2} - \sqrt{2}R_2J_{n_1n_2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.11. *If W_n is a wheel graph, then*

$$Ch[EDE(W_n)] = (\lambda + 3\sqrt{2})^{n-2}[\lambda^2 - 3\sqrt{2}(n-2)\lambda - (n-1)(9+(n-1)^2)]$$

Proof. The graph W_n of order n has two types of vertices namely, $n-1$ rim vertices are of degree 3 and central vertex has degree $n-1$. Hence,

$$EDE(W_n) = \begin{bmatrix} 3\sqrt{2}(J_{n-1} - I_{n-1}) & \sqrt{(9+(n-1)^2)}J_{(n-1)\times 1} \\ \sqrt{(9+(n-1)^2)}J_{1\times(n-1)} & \sqrt{2}(n-1)(J_1 - I_1) \end{bmatrix}.$$

$$Ch[EDE(W_n)] = \mid \lambda I - EDE(W_n) \mid$$

$$= \begin{vmatrix} (\lambda + 3\sqrt{2})I_{n-1} - 3\sqrt{2}J_{n-1} & -\sqrt{(9+(n-1)^2)}J_{(n-1)\times 1} \\ -\sqrt{(9+(n-1)^2)}J_{1\times(n-1)} & (\lambda + \sqrt{2}(n-1))I_1 - \sqrt{2}(n-1)J_1 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.12. *If F_t^3 is a friendship graph, then*

$$Ch[EDE(F_t^3)] = (\lambda + 2\sqrt{2})^{2t-1}[\lambda^2 - 2\sqrt{2}(2t-1)\lambda - 2t(4+(2t)^2)]$$

Proof. The graph F_t^3 of order $2t+1$ has two types of vertices namely, $2t$ vertices of degree 2 and one vertex of degree $2t$. Hence,

$$EDE(F_t^3) = \begin{bmatrix} 2\sqrt{2}(J_{2t} - I_{2t}) & \sqrt{(4+(2t)^2)}J_{2t\times 1} \\ \sqrt{(4+(2t)^2)}J_{1\times 2t} & 2\sqrt{2}t(J_1 - I_1) \end{bmatrix}.$$

$$Ch[F_t^3] = \mid \lambda I - EDE(F_t^3) \mid$$

$$= \begin{vmatrix} (\lambda + 2\sqrt{2})I_{2t} - 2\sqrt{2}J_{2t} & -\sqrt{(4+(2t)^2)}J_{2t\times 1} \\ -\sqrt{(4+(2t)^2)}J_{1\times 2t} & (\lambda + 2\sqrt{2}t)I_1 - 2\sqrt{2}tJ_1 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.13. *If $H_n - c$ is a helm without central vertex, then*

$$Ch[EDE(H_n - c)] = (\lambda + 3\sqrt{2})^{n-2}(\lambda + \sqrt{2})^{n-2}[\lambda^2 - 4\sqrt{2}(n-2)\lambda + 6(n-2)^2 - 10(n-1)^2$$

Proof. The graph $H_n - c$ of order $2(n-1)$ has two types of vertices namely, $n-1$ vertices are of degree 3 and remaining $(n-1)$ vertices has degree 1. Hence,

$$EDE(H_n - c) = \begin{bmatrix} 3\sqrt{2}(J_{n-1} - I_{n-1}) & \sqrt{10}J_{(n-1)\times(n-1)} \\ \sqrt{10}J_{(n-1)\times(n-1)} & \sqrt{2}(J_{n-1} - I_{n-1}) \end{bmatrix}.$$

$$Ch[EDE(H_n - c)] = \mid \lambda I - EDE(H_n - c) \mid$$

$$= \begin{vmatrix} (\lambda + 3\sqrt{2})I_{n-1} - 3\sqrt{2}J_{n-1} & -\sqrt{10}J_{(n-1)\times(n-1)} \\ -\sqrt{10}J_{(n-1)\times(n-1)} & (\lambda + \sqrt{2})I_{n-1} - \sqrt{2}J_{n-1} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.14. *If $H_n^{'} - c$ is a closed helm without central vertex, then*

$$Ch[EDE(H_n^{'} - c)] = (\lambda - 3\sqrt{2}(2n-3))(\lambda + 3\sqrt{2})^{2n-3}$$

Proof. The closed helm without central vertex $H_n^{'} - c$ is 3-regular graph with $2(n-1)$ vertices. Hence the result follows from equation (4). □

Theorem 3.15. *If $SF_n - c$ is a sun flower graph without central vertex, then*

$$Ch[EDE(SF_n - c)] = (\lambda + 3\sqrt{2})^{n-2}(\lambda + 2\sqrt{2})^{n-2}[\lambda^2 - 5\sqrt{2}(n-2)\lambda + 12(n-2)^2 - 13(n-1)^2]$$

Proof. The sun flower graph $SF_n - c$ without central vertex is a graph of order $2(n-1)$, which has two types of vertices. The $(n-1)$ vertices have degree 3 and the remaining $(n-1)$ vertices have degree 2. Hence,

$$EDE(SF_n - c) = \begin{bmatrix} 3\sqrt{2}(J_{n-1} - I_{n-1}) & \sqrt{13}J_{(n-1)\times(n-1)} \\ \sqrt{13}J_{(n-1)\times(n-1)} & 2\sqrt{2}(J_{n-1} - I_{n-1}) \end{bmatrix}.$$

$$Ch[EDE(SF_n - c)] = \mid \lambda I - EDE(SF_n - c) \mid$$

$$= \begin{vmatrix} (\lambda + 3\sqrt{2})I_{n-1} - 3\sqrt{2}J_{n-1} & -\sqrt{13}J_{(n-1)\times(n-1)} \\ -\sqrt{13}J_{(n-1)\times(n-1)} & (\lambda + 2\sqrt{2})I_{n-1} - 2\sqrt{2}J_{n-1} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.16. *If DC_n is a double cone, then,*

$$Ch[EDE(C_n)] = (\lambda + 4\sqrt{2})^{n-1}(\lambda + \sqrt{2}n)[\lambda^2 - (\sqrt{2}n + 4\sqrt{2}(n-1))\lambda + 8n(n-1) - 40n]$$

Proof. The double cone is a graph of of order $(n+2)$ has two types of vertices. The n vertices have degree 4 and the remaining 2 vertices have degree n. Hence,

$$EDE(DC_n) = \begin{bmatrix} 4\sqrt{2}(J_n - I_n) & 2\sqrt{5}J_{n\times 2} \\ 2\sqrt{5}J_{2\times n} & n\sqrt{2}(J_2 - I_2) \end{bmatrix}.$$

$$Ch[EDE(DC_n)] = \mid \lambda I - EDE(DC_n) \mid$$

$$= \begin{vmatrix} (\lambda + 4\sqrt{2})I_n - 4\sqrt{2}J_n & -2\sqrt{5}J_{n\times 2} \\ -2\sqrt{5}J_{2\times n} & (\lambda + n\sqrt{2})I_2 - n\sqrt{2}J_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.17. *If B_b is a book graph, then*

$$Ch[EDE(B_b)] = (\lambda + 2\sqrt{2})^{2b-1}(\lambda + \sqrt{2}(b+1))[\lambda^2 - (\sqrt{2}(b+1) + 2\sqrt{2}(2b-1))\lambda + 4(2b-1)(b+1) - 4b(4 + (b+1)^2)]$$

Proof. The Book graph B_b of order $(2b+2)$ has two types of vertices. The $2b$ vertices with degree 2 and 2 vertices are with degree $(b+1)$. Hence,

$$EDE(B_b) = \begin{bmatrix} 2\sqrt{2}(J_{2b} - I_{2b}) & \sqrt{4 + (b+1)^2}J_{2b\times 2} \\ \sqrt{4 + (b+1)^2}J_{2\times 2b} & \sqrt{2}(b+1)(J_2 - I_2) \end{bmatrix}.$$

$$Ch[EDE(B_b)] = \mid \lambda I - EDE(B_b) \mid$$

$$= \begin{vmatrix} (\lambda + 2\sqrt{2})I_{2b} - 2\sqrt{2}J_{2b} & -\sqrt{4 + (b+1)^2}J_{2b\times 2} \\ -\sqrt{4 + (b+1)^2}J_{2\times 2b} & (\lambda + \sqrt{2}(b+1))I_2 - \sqrt{2}(b+1)J_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.18. *If B_t is a book with triangular pages, then*

$$Ch[EDE(B_t)] = (\lambda + 2\sqrt{2})^{t-1}(\lambda + \sqrt{2}(t+1))[\lambda^2 - (\sqrt{2}(t+1) + 2\sqrt{2}(t-1))\lambda + 4(t-1)(t+1) - 2t(4 + (t+1)^2)]$$

Proof. The book B_t with triangular pages of order $(t+2)$ has two types of vertices. The t vertices have degree 2 and the remaining 2 vertices have degree $(t+1)$. Hence,

$$EDE(B_t) = \begin{bmatrix} 2\sqrt{2}(J_t - I_t) & \sqrt{4+(t+1)^2}J_{t\times 2} \\ \sqrt{4+(t+1)^2}J_{2\times t} & \sqrt{2}(t+1)(J_2 - I_2) \end{bmatrix}.$$

$$Ch[EDE(B_t)] = |\,\lambda I - EDE(B_t)\,|$$
$$= \begin{vmatrix} (\lambda + 2\sqrt{2})I_t - 2\sqrt{2}J_t & -\sqrt{4+(t+1)^2}J_{t\times 2} \\ -\sqrt{4+(b+1)^2}J_{2\times t} & (\lambda + \sqrt{2}(t+1))I_2 - \sqrt{2}(t+1)J_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.19. *If L_n is a ladder graph, then*

$$Ch[EDE(L_n)] = (\lambda + 3\sqrt{2})^{2n-5}(\lambda + 2\sqrt{2})^3[\lambda^2 - (3\sqrt{2}(2n-5) + 6\sqrt{2})\lambda + 36(2n-5) - 52(2n-4)$$

Proof. The ladder graph L_n is a graph of order $2n$ and has two types of vertices. The four vertices of degree 2 and $(2n-4)$ vertices of degree 3. Hence,

$$EDE(L_n) = \begin{bmatrix} 3\sqrt{2}(J_{2n-4} - I_{2n-4}) & \sqrt{13}J_{(2n-4)\times 4} \\ \sqrt{13}J_{4\times(2n-4)} & 2\sqrt{2}(J_4 - I_4) \end{bmatrix}.$$

$$Ch[EDE(L_n)] = |\,\lambda I - EDE(L_n)\,|$$
$$= \begin{vmatrix} (\lambda + 3\sqrt{2})I_{2n-4} - 9J_{2n-4} & -\sqrt{13}J_{(2n-4)\times 4} \\ -\sqrt{13}J_{4\times(2n-4)} & (\lambda + 2\sqrt{2})I_4 - 2\sqrt{2}J_4 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.20. *If Pr_n is a prism graph, then*

$$Ch[EDE(Pr_n)] = (\lambda + 3\sqrt{2})^{2n-1}(\lambda - 3\sqrt{2}(2n-1))$$

Proof. The prism Pr_n is 3-regular graph with $2n$ vertices. Hence, the result follows from equation (4).

□

Theorem 3.21. *If T_n is a triangular snake, then*

$$Ch[EDE(T_n)] = (\lambda + 2\sqrt{2})^n(\lambda + 4\sqrt{2})^{n-3}[\lambda^2 - (4\sqrt{2}(n-3) + 2\sqrt{2}n)\lambda + 16n(n-3) - 20(n+1)(n-2)]$$

Proof. The triangular snake T_n of order $(2n-1)$ has two types of vertices. The $(n+1)$ vertices have degree 2 and the remaining $(n-2)$ vertices have degree 4. Hence,

$$EDE(T_n) = \begin{bmatrix} 2\sqrt{2}(J_{n+1} - I_{n+1}) & 2\sqrt{5}J_{(n+1)\times(n-2)} \\ 2\sqrt{5}J_{(n-2)\times(n+1)} & 4\sqrt{2}(J_{n-2} - I_{n-2}) \end{bmatrix}.$$

$$Ch[EDE(T_n)] = |\,\lambda I - EDE(T_n)\,|$$
$$= \begin{vmatrix} (\lambda + 2\sqrt{2})I_{n+1} - 2\sqrt{2}J_{n+1} & -2\sqrt{5}J_{(n+1)\times(n-2)} \\ -2\sqrt{5}J_{(n-2)\times(n+1)} & (\lambda + 4\sqrt{2})I_{n-2} - 4\sqrt{2}J_{n-2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.22. *If Q_n is a quadrilateral snake, then*

$$Ch[EDE(Q_n)] = (\lambda + 2\sqrt{2})^{2n-1}(\lambda + 4\sqrt{2})^{n-3}[\lambda^2 - (4\sqrt{2}(n-3) + 2\sqrt{2}(2n-1))\lambda + 16(2n-1)(n-3) - 40n(n-2)]$$

Proof. The quadrilateral snake Q_n of order $3n-2$ has two types of vertices. The $2n$ vertices have degree 2 and the remaining $(n-2)$ vertices have degree 4. Hence,

$$EDE(Q_n) = \begin{bmatrix} 2\sqrt{2}(J_{2n} - I_{2n}) & 2\sqrt{5}J_{(2n)\times(n-2)} \\ 2\sqrt{5}J_{(n-2)\times(2n)} & 4\sqrt{2}(J_{n-2} - I_{n-2}) \end{bmatrix}.$$

$$Ch[EDE(Q_n)] = \mid \lambda I - EDE(Q_n) \mid$$

$$= \begin{vmatrix} (\lambda + 2\sqrt{2})I_{2n} - 2\sqrt{2}J_{2n} & -2\sqrt{5}J_{(2n)\times(n-2)} \\ -2\sqrt{5}J_{(n-2)\times(2n)} & (\lambda + 4\sqrt{2})I_{n-2} - 4\sqrt{2}J_{n-2} \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.23. *If G is an r-regular graph of order n, then*

$$Ch[EDE(J(G))] = (\lambda + \sqrt{2}r_1(\frac{nr}{2} - 1))(\lambda - \sqrt{2}r_1)^{(\frac{nr}{2}-1)} \quad where, \quad r_1 = \frac{(n-4)r}{2} + 1$$

Proof. The jump graph $J(G)$ is r-regular graph is $r_1 = (\frac{(n-4)r}{2} + 1)$-regular graph with $\frac{nr}{2}$ vertices. Hence, the result follows from equation (4). □

Theorem 3.24. *If S_n is a Star graph, then*

$$Ch[EDE(S_n)] = (\lambda + 1)^{n-2}[\lambda^2 - (n-2)\lambda - \frac{(n-1)(1 + (n-1)^2)}{4}]$$

Proof. The graph S_n of order n has two types of vertices namely, $(n-1)$ vertices are of degree 1 and central vertex has degree $(n-1)$. Hence,

$$EDE(S_n) = \begin{bmatrix} \sqrt{2}(J_{n-1} - I_{n-1}) & \sqrt{1 + (n-1)^2}J_{(n-1)\times 1} \\ \sqrt{1 + (n-1)^2}J_{1\times(n-1)} & \sqrt{2}(n-1)(J_1 - I_1) \end{bmatrix}.$$

$$Ch[EDE(S_n)] = \mid \lambda I - EDE(S_n) \mid$$

$$= \begin{vmatrix} (\lambda + \sqrt{2})I_{n-1} - \sqrt{2}J_{n-1} & -\sqrt{1 + (n-1)^2}J_{(n-1)\times 1} \\ -\sqrt{1 + (n-1)^2}J_{1\times(n-1)} & (\lambda + \sqrt{2}(n-1))I_1 - \sqrt{2}(n-1)J_1 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.25. *If $S_{n,n}$ is a double star graph, then*

$$Ch[EDE(S_{n,n})] = (\lambda + \sqrt{2})^{2n-3}(\lambda + \sqrt{2}n)[\lambda^2 - (\sqrt{2}(2n-3) + n\sqrt{2})\lambda + 2n(2n-3) - 4(n-1)(n^2+1)]$$

Proof. The graph $S_{n,n}$ of order $2n$ has two types of vertices namely, $(2n-1)$ vertices are of degree 1 and remaining two of degree n. Hence,

$$EDE(S_{n,n}) = \begin{bmatrix} \sqrt{2}(J_{2n-2} - I_{2n-2}) & \sqrt{n^2+1}J_{(2n-2)\times 2} \\ \sqrt{n^2+1}J_{2\times(2n-2)} & \sqrt{2}n(J_2 - I_2) \end{bmatrix}.$$

$$Ch[EDE(S_{n,n})] = \mid \lambda I - EDE(S_{n,n}) \mid$$

$$= \begin{vmatrix} (\lambda + \sqrt{2})I_{2n-2} - \sqrt{2}J_{2n-2} & -\sqrt{(n^2+1)}J_{(2n-2)\times 2} \\ -\sqrt{(n^2+1)}J_{2\times(2n-2)} & (\lambda + \sqrt{2}n)I_2 - \sqrt{2}nJ_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.26. *If $K_{m,n}$ is a complete bipartite graph, then*

$$Ch[EDE(K_{m,n})] = (\lambda + \sqrt{2}n)^{m-1}(\lambda + \sqrt{2}m)^{n-1}[\lambda^2 - (\sqrt{2}m(n-1) + \sqrt{2}n(m-1))\lambda \\ +2(m-1)(n-1)mn - mn(m^2+n^2)]$$

Proof. The graph $K_{m,n}$ of order $(m+n)$ has two types of vertices namely, m vertices are of degree n and n vertices of degree m. Hence,

$$EDE(K_{m,n}) = \begin{bmatrix} \sqrt{2}n(J_m - I_m) & \sqrt{m^2+n^2}J_{m\times n} \\ \sqrt{m^2+n^2}J_{n\times m} & \sqrt{2}m(J_n - I_n) \end{bmatrix}.$$

$$Ch[EDE(K_{m,n})] = \mid \lambda I - EDE(K_{m,n}) \mid$$

$$= \begin{vmatrix} (\lambda + \sqrt{2}n)I_m - \sqrt{2}nJ_m & -\sqrt{m^2+n^2}J_{m\times n} \\ -\sqrt{m^2+n^2}J_{m\times n} & (\lambda + \sqrt{2}m)I_n - \sqrt{2}mJ_n \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

Theorem 3.27. *If P_n is a path graph, then*

$$Ch[EDE(P_n)] = (\lambda + 2\sqrt{2})^{n-3}(\lambda + \sqrt{2})[\lambda^2 - (2\sqrt{2}(n-3) + \sqrt{2})\lambda + 4(n-3) - 10(n-2)]$$

Proof. The graph P_n of order n has two types of vertices namely, $(n-2)$ vertices are of degree 2 and remaining two end vertices of degree 1. Hence,

$$EDE(P_n) = \begin{bmatrix} 2\sqrt{2}(J_{n-2} - I_{n-2}) & \sqrt{5}J_{(n-2)\times 2} \\ \sqrt{5}J_{2\times(n-2)} & \sqrt{2}(J_2 - I_2) \end{bmatrix}.$$

$$Ch[EDE(P_n)] = \mid \lambda I - EDE(P_n) \mid$$

$$= \begin{vmatrix} (\lambda + 2\sqrt{2})I_{n-2} - 2\sqrt{2}J_{n-2} & -\sqrt{5}J_{(n-2)\times 2} \\ -\sqrt{5}J_{2\times(n-2)} & (\lambda + \sqrt{2})I_2 - \sqrt{2}J_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

A dumbbell is the graph obtained from two disjoint cycles by joining them by a path.

Theorem 3.28. *If $D_{n,n}$ is a dumbbell graph, then*

$$Ch[EDE(D_{n,n})] = (\lambda + 2\sqrt{2})^{2n-3}(\lambda + 3\sqrt{2})[\lambda^2 - (2\sqrt{2}(2n-3) + 3\sqrt{2})\lambda + 12(2n-3) - 52(n-1)]$$

Proof. The graph $D_{n,n}$ of order $2n$ has two types of vertices namely, $2n-2$ vertices are of degree 2 and remaining two of degree 3. Hence,

$$EDE(D_{n,n}) = \begin{bmatrix} 2\sqrt{2}(J_{2n-2} - I_{2n-2}) & \sqrt{13}J_{(2n-2)\times 2} \\ \sqrt{13}J_{2\times(2n-2)} & 3\sqrt{2}(J_2 - I_2) \end{bmatrix}.$$

$$Ch[EDE(D_{n,n})] = \mid \lambda I - EDE(D_{n,n}) \mid$$

$$= \begin{vmatrix} (\lambda + 2\sqrt{2})I_{2n-2} - 2\sqrt{2}J_{2n-2} & -\sqrt{13}J_{(2n-2)\times 2} \\ -\sqrt{13}J_{2\times(2n-2)} & (\lambda + 3\sqrt{2})I_2 - 3\sqrt{2}J_2 \end{vmatrix}.$$

Now by using Lemma 2.21, we get the desired result. □

4. Hyperenergetic graphs

A graph G with n vertices is said to be hyperenergetic [11] if $\mathcal{E}(G) \geq 2n-2$, and to be nonhyperenergetic if $\mathcal{E}(G) \leq 2n-2$. A noncomplete graph whose energy is equal to $(2n-2)$ is called borderenergetic [9].

Definition 4.1. A graph G of order n is said to be Euclidean degree hyperenergetic if $EDE(G) \geq 2\sqrt{2}(n-1)^2$.

Definition 4.2. A graph G of order n is said to be Euclidean degree nonhyperenergetic if $EDE(G) \leq 2\sqrt{2}(n-1)^2$.

Definition 4.3. A noncomplete graph of order n whose energy is equal to $2\sqrt{2}(n-1)^2$ is called Euclidean degree borderenergetic.

Definition 4.4. Two graphs G_1 and G_2 are said to be Euclidean degree equienergetic if they have same Euclidean degree energy. That is, $\mathcal{E}[EDE(G_1)] = \mathcal{E}[EDE(G_2)]$.

Theorem 4.5. *If G is an r-regular graph of order n, then $\overline{G}$ is*

(i) Euclidean degree borderenergetic for $r = 0$,

(ii) Euclidean degree nonhyperenergetic for $r \geq 1$.

Proof. The graph $\overline{G}$ is $(n-1-r)$-regular graph.

$$Ch[EDE(\overline{G}), \lambda] = (\lambda - \sqrt{2}(n-1)(n-1-r))(\lambda + \sqrt{2}(n-1-r))^{n-1}$$

Thus ,

$$\mathcal{E}[EDE(\overline{G})] = 2\sqrt{2}(n-1-r)(n-1)$$

From Definition 4.1, the graph G is Euclidean degree hyperenergetic if $\mathcal{E}(\overline{G}) > 2\sqrt{2}(n-1)^2$.
That is, if $2\sqrt{2}(n-1-r)(n-1) \geq 2\sqrt{2}(n-1)^2$. This inequality does not hold for any value of r, whereas the two quantities are equal when $r = 0$. Hence, $\overline{G}$ is Euclidean degree borderenergetic for $r = 0$ and Euclidean degree nonhyperenergetic for $r \geq 1$. □

Theorem 4.6. *The graph $L(K_n)$ is Euclidean degree borderenergetic for $n = 2, 3$ and Euclidean degree nonhyperenergetic for $n \geq 4$.*

Proof. The complete graph K_n is an $(n-1)$-regular graph of order n. Thus,

$$Ch[EDE(K_n), \lambda] = (\lambda - \sqrt{2}(n-1)^2)(\lambda + \sqrt{2}(n-1))^{n-1}$$

The line graph of K_n is $L(K_n)$ is an $(2n-4)$-regular graph of order $n_1 = \frac{nr}{2}$ and,

$$Ch[EDE(K_n), \lambda] = (\lambda - 2\sqrt{2}(n-2)(\frac{nr}{2} - 1))(\lambda + 2\sqrt{2}(n-2))^{\frac{nr}{2}-1}$$

Hence,

$$E[EDE(L(K_n))] = 2\sqrt{2}(n-2)(nr-2)$$

Clearly, $\mathcal{E}[EDE(L(K_n))] \leq 2\sqrt{2}(\frac{n(n-1)}{2} - 1)^2$ for $n \geq 4$ and equality holds for $n = 2, 3$.
Hence, $L(K_2)$, $L(K_3)$ are Euclidean degree borderenergetic and $L(K_n)$ is Euclidean degree nonhyperenergetic for $n \geq 4$. □

Theorem 4.7. *If G is an r-regular graph of order n, then $J(G)$ is*

(i) Euclidean degree borderenergetic for $r = 1$,

(ii) Euclidean degree nonhyperenergetic for $r \geq 2$.

Proof. The jump graph $J(G)$ of the r-regular graph G is $r_1 = (\frac{(n-4)r}{2} + 1)$-regular graph with $\frac{nr}{2}$ vertices.

$$Ch[EDE(J(G))] = (\lambda + \sqrt{2}r_1(\frac{nr}{2} - 1))(\lambda - \sqrt{2}r_1)^{(\frac{nr}{2}-1)} \quad where, \quad r_1 = \frac{(n-4)r}{2} + 1$$

Hence,

$$\mathcal{E}[EDE(J(G)] = 2\sqrt{2}r_1(\frac{nr}{2} - 1)$$
$$= \sqrt{2}((n-4)r + 2)(\frac{nr}{2} - 1)$$

$\mathcal{E}[EDE(J(G))] \leq 2\sqrt{2}(\frac{nr}{2} - 1)^2$ for $r \geq 2$ and equality holds for $r = 1$. □

Theorem 4.8. *If G is an r-regular graph of order n, then $T(G)$ is Euclidean degree nonhyperenergetic.*

Proof. The total graph $T(G)$ of an r-regular graph G is a regular graph of degree $2r$ with $n + \frac{nr}{2}$ vertices. Then,

$$Ch[EDE(T(G))] = (\lambda - 2\sqrt{2}r(n + \frac{nr}{2} - 1))(\lambda + 2\sqrt{2}r)^{n+\frac{nr}{2}-1}$$

Hence,

$$\mathcal{E}(EDE(T(G))) = 4\sqrt{2}r(n + \frac{nr}{2} - 1)$$

$\mathcal{E}[EDE(T(G))] \leq 2\sqrt{2}(n + \frac{nr}{2} - 1)^2$ for all r. Thus $T(G)$ is Euclidean degree nonhyperenergetic. □

5. Conclusion

We conclude with the following observations.
In this paper, we have obtained the characteristic polynomial of the Euclidean degree matrix of graphs obtained by some graphs operations. Also, bounds for both largest Euclidean degree eigenvalue and Euclidean degree energy of graphs are established. we have characterized Euclidean degree hyperenergetic, borderenergetic and equienergetic of some graphs.

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