

11. Proximal and Quasi-Proximal Relators

Analogously to Definition 10.1, we may also naturally have the following

Definition 11.1. A relator \mathcal{R} on X is called

- (1) *quasi-proximal* if $A \in \text{Int}_{\mathcal{R}}[\tau_{\mathcal{R}} \cap \text{Int}_{\mathcal{R}}(R[A])]$ for all $A \subseteq X$ and $R \in \mathcal{R}$;
- (2) *proximal* if for any $A \subseteq X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[A]$.

Remark 11.2. Thus, the relator \mathcal{R} is quasi-proximal if and only if, for any $A \subseteq X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $A \in \text{Int}_{\mathcal{R}}(V)$ and $V \in \text{Int}_{\mathcal{R}}(R[A])$.

Now, by using the corresponding definitions, we can also easily prove the following analogues of Theorems 10.3 and 10.5.

Theorem 11.3. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is proximal ;
- (2) \mathcal{R} is reflexive and quasi-proximal .

Proof. To prove the implication (1) \implies (2), note that if (1) holds, then for any $A \subseteq X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[A]$. Hence, since A may be $\{x\}$ for any $x \in X$, and $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can at once see that \mathcal{R} is not only reflexive, but also topological.

Moreover, since $V \in \tau_{\mathcal{R}}$, we can also note $V \in \text{Int}_{\mathcal{R}}(V)$. Hence, by using that $A \subseteq V$ and $V \subseteq R[A]$, we can already infer that the inclusions $A \in \text{Int}_{\mathcal{R}}(V)$ and $V \in \text{Int}_{\mathcal{R}}(R[A])$ are also true. Therefore, by Remark 11.2, \mathcal{R} is quasi-proximal, and thus (2) also holds. \square

Remark 11.4. Note that if \mathcal{R} is only a weakly proximal relator on X in the sense that, for any $x \in X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$, then because of $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$ we can already state that \mathcal{R} is topological.

Theorem 11.5. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is proximal ;
- (2) $\text{Int}_{\mathcal{R}}(A) = \bigcup \{ \mathcal{P}(V) : A \supseteq V \in \tau_{\mathcal{R}} \}$ for all $A \subseteq X$;
- (3) $\text{Cl}_{\mathcal{R}}(A) = \bigcap \{ \mathcal{P}(X) \setminus \mathcal{P}(W^c) : A \subseteq W \in \tau_{\mathcal{R}} \}$ for all $A \subseteq X$.

Proof. To check the equivalence of (1) and (2), note that, for any $A, B \subseteq X$, we have $B \in \bigcup \{ \mathcal{P}(V) : A \supseteq V \in \tau_{\mathcal{R}} \}$ if and only if there exists $V \in \tau_{\mathcal{R}}$ such that $V \subseteq A$ and $B \subseteq V$. Thus, $\bigcup \{ \mathcal{P}(V) : A \supseteq V \in \tau_{\mathcal{R}} \} \subseteq \text{Int}_{\mathcal{R}}(A)$ is always true.

Moreover, if $A \subseteq X$ and $R \in \mathcal{R}$, then because of $R[A] \subseteq R[A]$, we always have $A \in \text{Int}_{\mathcal{R}}(R[A])$. Therefore, if the essential part of (2) holds, then there exists $V \in \tau_{\mathcal{R}}$ such that $V \subseteq R[A]$ and $A \subseteq V$, and thus (1) also holds. \square

Now, since $\mathcal{P}(A) = \{B : B \subseteq A\}$ and $\mathcal{P}(A) = \text{Int}_{\Delta_X}(A)$, we can also state

Corollary 11.6. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is proximal ;
- (2) $\text{Int}_{\mathcal{R}}(A) = \mathcal{P}[\tau_{\mathcal{R}} \cap \mathcal{P}(A)]$ for all $A \subseteq X$;
- (3) $\text{Int}_{\mathcal{R}}(A) = \text{Int}_{\Delta_X}[\tau_{\mathcal{R}} \cap \text{Int}_{\Delta_X}(A)]$ for all $A \subseteq X$.

However, it is now more important to note that we also have the following

Theorem 11.7. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is proximal ;
- (2) \mathcal{R} is proximally equivalent to \mathcal{R}^{∞} or $\mathcal{R}^{\#\infty}$;
- (3) \mathcal{R} is proximally equivalent to a preorder relator on X .

In principle, each theorem on topological and quasi-topological relators can be immediately derived from a corresponding theorem on proximal and quasi-proximal relators by using the following two theorems.

Theorem 11.8. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is quasi-topological; (2) \mathcal{R}^\wedge is quasi-proximal.

Proof. To prove the implication (1) \implies (2), assume that (1) holds, and moreover $A \subseteq X$ and $S \in \mathcal{R}^\wedge$. Define $V = \text{int}_{\mathcal{R}}(S[A])$. Then, if $\mathcal{R} \neq \emptyset$, by Theorem 10.2 and Corollary 7.3, we have $V \in \mathcal{T}_{\mathcal{R}} = \tau_{\mathcal{R}^\wedge}$. Moreover, since $V \subseteq \text{int}_{\mathcal{R}}(S[A])$, by Theorem 7.1 we also have $V \in \text{Int}_{\mathcal{R}^\wedge}(S[A])$. Therefore, $V \in \tau_{\mathcal{R}^\wedge} \cap \text{Int}_{\mathcal{R}^\wedge}(S[A])$.

On the other hand, since $S \in \mathcal{R}^\wedge$ and $S[A] \subseteq S[A]$, we can also note that $A \in \text{Int}_{\mathcal{R}^\wedge}(S[A])$. Hence, by Theorem 7.1, we can infer that $A \subseteq \text{int}_{\mathcal{R}}(S[A]) = V$. Moreover, since $V \in \tau_{\mathcal{R}^\wedge}$, we can also note that $V \in \text{Int}_{\mathcal{R}^\wedge}(V)$. Hence, since $A \subseteq V$, we can infer that $A \in \text{Int}_{\mathcal{R}^\wedge}(V)$. Therefore, since $V \in \tau_{\mathcal{R}^\wedge} \cap \text{Int}_{\mathcal{R}^\wedge}(S[A])$, we also have $A \in \text{Int}_{\mathcal{R}^\wedge}[\tau_{\mathcal{R}^\wedge} \cap \text{Int}_{\mathcal{R}^\wedge}(S[A])]$. This shows that (2) also holds.

Thus, to complete the proof of the implication (1) \implies (2), it remains only to note that if $\mathcal{R} = \emptyset$, then \mathcal{R} is topological. Moreover, $\mathcal{R}^\wedge = \emptyset$ if $X \neq \emptyset$ and $\mathcal{R}^\wedge = \{\emptyset\}$ if $X = \emptyset$. Thus, \mathcal{R}^\wedge is proximal. \square

Remark 11.9. If assertion (2) holds, then \mathcal{R}^\wedge is semi-proximal in the sense that $A \in \text{Int}_{\mathcal{R}^\wedge}[\text{Int}_{\mathcal{R}^\wedge}(S[A])]$ for all $A \subseteq X$ and $S \in \mathcal{R}^\wedge$.

Moreover, if in particular $\{x\} \in \text{Int}_{\mathcal{R}^\wedge}[\text{Int}_{\mathcal{R}^\wedge}(R(x))]$ for all $x \in X$ and $R \in \mathcal{R}$, then we can already prove that assertion (1) also holds.

From Theorem 11.8, by using Theorems 10.3 and 11.3, we can easily derive

Theorem 11.10. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is topological, (2) \mathcal{R}^Δ is proximal.

Remark 11.11. By the corresponding definitions, it is clear that the relator \mathcal{R}^Δ is reflexive if and only if \mathcal{R} is reflexive.

However, if $\mathcal{R} \not\subseteq \{X^2\}$, then there exists $R \in \mathcal{R}$ such that $R \neq X^2$. Therefore, there exist $x, y \in X$ such that $x \notin R(y)$. Thus, $S = \{x\} \times R(y) \cup \{x\}^c \times X$ is a non-reflexive relation on X such that $S \in \mathcal{R}^\Delta$. Therefore, \mathcal{R}^Δ cannot be reflexive.

Note that if $\mathcal{R} = \emptyset$, then by Definition 6.1 we have $\mathcal{R}^\Delta = \emptyset$ if $X \neq \emptyset$, and $\mathcal{R}^\Delta = \{\emptyset\}$ if $X = \emptyset$. Moreover, if $\mathcal{R} = \{X^2\}$, then we have $\mathcal{R}^\Delta = \{X^2\}$. Therefore, in these very particular cases, the relator \mathcal{R}^Δ is also reflexive.

12. A Few Basic Facts on Filtered Relators

Intersection properties of relators were also first investigated in [146, 148].

Definition 12.1. A relator \mathcal{R} on X to Y is called

- (1) *properly filtered* if for any $R, S \in \mathcal{R}$ we have $R \cap S \in \mathcal{R}$;
- (2) *uniformly filtered* if for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subseteq R \cap S$;
- (3) *proximally filtered* if for any $A \subseteq X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T[A] \subseteq R[A] \cap S[A]$;
- (4) *topologically filtered* if for any $x \in X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T(x) \subseteq R(x) \cap S(x)$.

Remark 12.2. By using the binary operation \wedge and the unary operations $*$, $\#$ and \wedge , the above properties can be reformulated in some more concise forms.

For instance, we can see that \mathcal{R} is topologically filtered if and only if any one of the properties $\mathcal{R} \wedge \mathcal{R} \subseteq \mathcal{R}^\wedge$, $(\mathcal{R} \wedge \mathcal{R})^\wedge = \mathcal{R}^\wedge$ and $\mathcal{R}^\wedge \wedge \mathcal{R}^\wedge = \mathcal{R}^\wedge$ holds.

However, in general, we only have $(R \cap S)[A] \subseteq R[A] \cap S[A]$. Therefore, the corresponding proximal filteredness properties are, unfortunately, not equivalent.

Despite this, we can easily prove the following theorem which shows the appropriateness of the above proximal filteredness property.

Theorem 12.3. For a relator \mathcal{R} on X to Y , the following assertions are equivalent :

- (1) \mathcal{R} is proximally filtered ;
- (2) $\text{Cl}_{\mathcal{R}}(A \cup B) = \text{Cl}_{\mathcal{R}}(A) \cup \text{Cl}_{\mathcal{R}}(B)$ for all $A, B \subseteq Y$;
- (3) $\text{Int}_{\mathcal{R}}(A \cap B) = \text{Int}_{\mathcal{R}}(A) \cap \text{Int}_{\mathcal{R}}(B)$ for all $A, B \subseteq Y$.

Proof. To prove the implication (3) \implies (1), note that if $A \subseteq X$ and $R, S \in \mathcal{R}$, then by the definition of $\text{Int}_{\mathcal{R}}$ we trivially have $A \in \text{Int}_{\mathcal{R}}(R[A])$ and $A \in \text{Int}_{\mathcal{R}}(S[A])$. Therefore, if (3) holds, then we also have $A \in \text{Int}_{\mathcal{R}}(R[A] \cap S[A])$. Thus, by the definition of $\text{Int}_{\mathcal{R}}$, there exists $T \in \mathcal{R}$ such that $T[A] \subseteq R[A] \cap S[A]$. \square

Now, as an immediate consequence of this theorem, we can also state

Corollary 12.4. If \mathcal{R} is a proximally filtered relator on X , then the families $\varepsilon_{\mathcal{R}}$ and $\tau_{\mathcal{R}}$ are closed under binary unions and intersections, respectively.

From Theorem 12.3, we can also easily derive the following

Theorem 12.5. For a relator \mathcal{R} on X to Y , the following assertions are equivalent :

- (1) \mathcal{R} is topologically filtered ;
- (2) $\text{cl}_{\mathcal{R}}(A \cup B) = \text{cl}_{\mathcal{R}}(A) \cup \text{cl}_{\mathcal{R}}(B)$ for all $A, B \subseteq Y$;
- (3) $\text{int}_{\mathcal{R}}(A \cap B) = \text{int}_{\mathcal{R}}(A) \cap \text{int}_{\mathcal{R}}(B)$ for all $A, B \subseteq Y$.

Thus, in particular, we can also state the following

Corollary 12.6. If \mathcal{R} is a topologically filtered relator on X , then the families $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ are closed under binary unions and intersections, respectively.

The following example shows that, for a non-topological relator \mathcal{R} , the converse of the above corollary need not be true.

Example 12.7. If $X = \{1, 2, 3\}$ and R_i is relation on X , for each $i = 1, 2$, such that

$$R_1(1) = \{1, i+1\} \quad \text{and} \quad R_2(2) = R_2(3) = \{2, 3\},$$

then $\mathcal{R} = \{R_1, R_2\}$ is reflexive relator on X such that $\mathcal{T}_{\mathcal{R}}$ is closed under arbitrary intersections, but \mathcal{R} is still not topologically filtered.

By the corresponding definitions, it is clear that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, \{2, 3\}, X\}$. Moreover, we can note that $R_1(1) \not\subseteq R_1(1) \cap R_2(1)$ for each $i = 1, 2$, and thus by Definition 12.1 the relator \mathcal{R} is not topologically filtered.

13. A Few Basic Facts on Quasi-Filtered Relators

Since $R \subseteq R^{\infty}$ for every relation R on X , in addition to Definition 12.1, we may also naturally introduce the following

Definition 13.1. A relator \mathcal{R} on X is called

- (1) *quasi-uniformly filtered* if for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subseteq R^{\infty} \cap S^{\infty}$;
- (2) *quasi-proximally filtered* if for any $A \subseteq X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T[A] \subseteq R^{\infty}[A] \cap S^{\infty}[A]$;
- (3) *quasi-topologically filtered* if for any $x \in X$ and $R, S \in \mathcal{R}^{\wedge}$ there exists $T \in \mathcal{R}$ such that $T(x) \subseteq R^{\infty}(x) \cap S^{\infty}(x)$.

Remark 13.2. Analogously to Remark 12.2, the above quasi-filteredness properties can also be reformulated in some more concise forms.

For instance, we can see that \mathcal{R} is quasi-topologically filtered if and only if $\mathcal{R}^{\wedge\infty} \wedge \mathcal{R}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge}$, $(\mathcal{R}^{\wedge\infty} \wedge \mathcal{R}^{\wedge\infty})^{\wedge\infty} = \mathcal{R}^{\wedge\infty}$ or $\mathcal{R}^{\wedge\infty} \wedge \mathcal{R}^{\wedge\infty} = \mathcal{R}^{\wedge\infty}$.

However, it is now more important to note that, by using some former results, we can also prove the following two theorems which show the appropriateness of the above quasi-proximal and quasi-topological filteredness properties.

Theorem 13.3. *For any relator \mathcal{R} on X , the following assertions are equivalent :*

- (1) \mathcal{R} is a quasi-proximally filtered ;
- (2) $\tau_{\mathcal{R}}$ is closed under binary unions ;
- (3) $\tau_{\mathcal{R}}$ is closed under binary intersections .

Theorem 13.4. *For any relator \mathcal{R} on X , the following assertions are equivalent :*

- (1) \mathcal{R} is a quasi-topologically filtered ;
- (2) $\mathcal{F}_{\mathcal{R}}$ is closed under binary unions ;
- (3) $\mathcal{T}_{\mathcal{R}}$ is closed under binary intersections .

Remark 13.5. In this respect it is also worth mentioning that if \mathcal{R} is a relator on X to Y , then the family $\mathcal{E}_{\mathcal{R}}$ is closed under binary intersections if and only if \mathcal{R} is quasi-directed in the sense that for any $x, y \in X$ and $R, S \in \mathcal{R}$ we have $R(x) \cap S(y) \in \mathcal{E}_{\mathcal{R}}$.

From the above two theorems, by using Corollaries 12.4 and 12.6, we can derive

Corollary 13.6. *If \mathcal{R} is a proximally (topologically) filtered relator on X , then \mathcal{R} is also quasi-proximally (quasi-topologically) filtered.*

Now, by using Theorem 13.3, we can also easily prove the following

Theorem 13.7. *If \mathcal{R} is a quasi-proximally filtered, proximal relator on X , then \mathcal{R} is proximally filtered.*

Proof. Suppose that $A \subseteq X$ and $R, S \in \mathcal{R}$. Then, by Definition 11.1, there exist $U, V \in \tau_{\mathcal{R}}$ such that $A \subseteq U \subseteq R[A]$ and $A \subseteq V \subseteq S[A]$.

Moreover, by Theorem 13.3, we can state that $U \cap V \in \tau_{\mathcal{R}}$. Therefore, by the definition of $\tau_{\mathcal{R}}$, there exists $T \in \mathcal{R}$ such that $T[U \cap V] \subseteq U \cap V$. Hence, we can see that $T[A] \subseteq T[U \cap V] \subseteq U \cap V \subseteq R[A] \cap S[A]$. \square

Moreover, by using Theorem 13.4, we can quite similarly prove

Theorem 13.8. *If \mathcal{R} is a quasi-topologically filtered, topological relator on X , then \mathcal{R} is topologically filtered.*

Remark 13.9. Our former Example 12.7 shows that even a quasi-proximally filtered, reflexive relator need not be topologically filtered.

Namely, if X and \mathcal{R} are as in Example 12.7, then by the corresponding definitions it is clear that $\tau_{\mathcal{R}} = \{\emptyset, \{2, 3\}, X\}$, and thus by Theorem 13.3 the relator \mathcal{R} is quasi-proximally filtered.

14. Some Further Theorems on Topologically Filtered Relators

Now, by using the arguments of Kuratowski [88, pp. 39, 45], we shall prove some more particular theorems on the relation $\text{cl}_{\mathcal{R}}$.

To make the forthcoming proofs much shorter and more readable, we shall use the convenient notations

$$A^- = \text{cl}_{\mathcal{R}}(A), \quad A^\circ = \text{int}_{\mathcal{R}}(A) \quad \text{and} \quad A^\dagger = \text{res}_{\mathcal{R}}(A).$$

Theorem 14.1. *If \mathcal{R} is a topologically filtered relator on X to Y , then for any $A, B \subseteq Y$ we have*

$$\text{cl}_{\mathcal{R}}(A) \setminus \text{cl}_{\mathcal{R}}(B) = \text{cl}_{\mathcal{R}}(A \setminus B) \setminus \text{cl}_{\mathcal{R}}(B).$$

Proof. By using Theorem 12.5, we can see that

$$A^- \cup B^- = (A \cup B)^- = ((A \setminus B) \cup B)^- = (A \setminus B)^- \cup B^-.$$

Hence, we can already infer that

$$A^- \setminus B^- = (A^- \cup B^-) \setminus B^- = ((A \setminus B)^- \cup B^-) \setminus B^- = (A \setminus B)^- \setminus B^-.$$

Thus, in particular, we can also state the following □

Corollary 14.2. *If \mathcal{R} is a topologically filtered relator on X to Y , then for any $A, B \subseteq Y$ we have*

$$\text{cl}_{\mathcal{R}}(A) \setminus \text{cl}_{\mathcal{R}}(B) \subseteq \text{cl}_{\mathcal{R}}(A \setminus B).$$

This corollary will already allow us to easily prove the following

Theorem 14.3. *If \mathcal{R} is a topologically filtered relator on X , then for any $A \subseteq X$ and $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\text{cl}_{\mathcal{R}}(A) \cap U = \text{cl}_{\mathcal{R}}(A \cap U) \cap U.$$

Proof. By Definition 5.1 and Theorem 4.5, we have $U \subseteq U^{\circ} = U^{c-c}$. Hence, by using Corollary 14.2, we can already infer that

$$A^- \cap U \subseteq A^- \cap U^{c-c} = A^- \setminus U^{c-c} \subseteq (A \setminus U^c)^- = (A \cap U)^-.$$

Therefore, $A^- \cap U = A^- \cap U \cap U \subseteq (A \cap U)^- \cap U$.

Moreover, by using the increasingness of $-$, we can see that $(A \cap U)^- \subseteq A^-$, and thus $(A \cap U)^- \cap U \subseteq A^- \cap U$ is always true. Therefore, we actually have $A^- \cap U = (A \cap U)^- \cap U$. □

Thus, in particular, we can also state the following important corollary which can also be easily proved directly.

Corollary 14.4. *If \mathcal{R} is a topologically filtered relator on X , then for any $A \subseteq X$ and $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\text{cl}_{\mathcal{R}}(A) \cap U \subseteq \text{cl}_{\mathcal{R}}(A \cap U).$$

Remark 14.5. The importance of the closure space counterpart of Corollary 14.4 was also recognized Császár [33, 34, 35, 36, 38, 39] and Sivagami [140] who assumed it as an axiom for an increasing set-to-set function γ .

Moreover, it is also worth noticing that, by using Theorem 4.5, Corollary 14.4 can be reformulated in the dual form that if \mathcal{R} is a topologically filtered relator on X , then for any $A \subseteq X$ and $V \in \mathcal{F}_{\mathcal{R}}$ we have $\text{int}_{\mathcal{R}}(A \cup V) \subseteq \text{int}_{\mathcal{R}}(A) \cup V$.

15. Some More Particular Theorems on Topologically Filtered Relators

By using Corollary 14.4, we can also easily prove the following

Theorem 15.1. *If \mathcal{R} is a topologically filtered, topological relator on X , then for any $A \subseteq X$ and $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\text{cl}_{\mathcal{R}}(A \cap U) = \text{cl}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(A) \cap U).$$

Proof. By Corollary 14.4, we have $A^- \cap U \subseteq (A \cap U)^-$. Moreover, by Theorems 10.3 and 10.2, we have $(A \cap U)^- \in \mathcal{F}_{\mathcal{R}}$. Hence, by using the increasingness of $-$ and the definition $\mathcal{F}_{\mathcal{R}}$, we can infer that

$$(A^- \cap U)^- \subseteq (A \cap U)^{- -} \subseteq (A \cap U)^-.$$

On the other hand, by Theorems 10.3 and 9.3, we have $A \subseteq A^-$, and thus also $A \cap U \subseteq A^- \cap U$. Hence, by using the increasingness of $-$, we can infer that $(A \cap U)^- \subseteq (A^- \cap U)^-$. Therefore, the required equality $(A \cap U)^- = (A^- \cap U)^-$ is also true. □

From this theorem, we can immediately derive

Corollary 15.2. *If \mathcal{R} is a topologically filtered, topological relator on X , then for any $A \in \mathcal{D}_{\mathcal{R}}$ and $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\text{cl}_{\mathcal{R}}(U) = \text{cl}_{\mathcal{R}}(A \cap U).$$

Proof. By Definition 4.1 and Theorem 15.1, we evidently have

$$U^{-} = (X \cap U)^{-} = (A^{-} \cap U)^{-} = (A \cap U)^{-}.$$

Now, by modifying an argument of Levine [95], we can also prove □

Theorem 15.3. *If \mathcal{R} is a nonvoid, topological relator on X and $A \subseteq X$ such that $\text{cl}_{\mathcal{R}}(U) = \text{cl}_{\mathcal{R}}(A \cap U)$ for all $U \in \mathcal{T}_{\mathcal{R}}$, then $A \in \mathcal{D}_{\mathcal{R}}$.*

Proof. Assume on the contrary that $A \notin \mathcal{D}_{\mathcal{R}}$. Then, by Definition 4.1, there exists $x \in X$ such that $x \notin A^{-}$. Thus, by Definition 4.1, there exists $R \in \mathcal{R}$ such that $A \cap R(x) = \emptyset$. Moreover, by Definition 10.1, there exists $U \in \mathcal{T}_{\mathcal{R}}$ such that $x \in U \subseteq R(x)$. Thus, in particular we also have $A \cap U = \emptyset$.

Hence, by using the assumptions of the theorem, we can infer that

$$U^{-} = (A \cap U)^{-} = \emptyset^{-} = \emptyset.$$

Moreover, from the inclusion $x \in U$, by using Theorems 10.3 and 9.3 and the increasingness of $-$, we can infer that $x \in \{x\}^{-} \subseteq U^{-}$, and thus $U^{-} \neq \emptyset$. This contradiction proves that $A \in \mathcal{D}_{\mathcal{R}}$. □

Remark 15.4. If \mathcal{R} is a nonvoid, reflexive relator on X and $A \subseteq X$ such that $\text{cl}_{\mathcal{R}}(R(x)) = \text{cl}_{\mathcal{R}}(A \cap R(x))$ for all $x \in X$ and $R \in \mathcal{R}$, then we can even more easily prove that $A \in \mathcal{D}_{\mathcal{R}}$.

In addition to Theorem 15.3, we can also prove the following

Theorem 15.5. *If \mathcal{R} is a topologically filtered, topological relator on X , then for any $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\text{res}_{\mathcal{R}}(U) \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}.$$

Proof. By Theorem 5.5, we have $U^c \in \mathcal{F}_{\mathcal{R}}$. Moreover, by Theorems 10.3 and 10.2, we also have $U^{-} \in \mathcal{F}_{\mathcal{R}}$. Hence, by using Corollary 12.6, we can already infer that

$$U^{\dagger} = U^{-} \setminus U = U^{-} \cap U^c \in \mathcal{F}_{\mathcal{R}}.$$

Moreover, by using Theorems 12.5 and 9.3 and the increasingness of $-$, we can also see that

$$U^{\dagger \circ} = (U^{-} \setminus U)^{\circ} = (U^{-} \cap U^c)^{\circ} = U^{-\circ} \cap U^{c\circ} = U^{-\circ} \cap U^{-c} \subseteq U^{-} \cap U^{-c} = \emptyset,$$

and thus $U^{\dagger \circ} = \emptyset$. Therefore, $U^{\dagger} \notin \mathcal{E}_{\mathcal{R}}$, and thus $U^{\dagger} \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$. □

Now, as an immediate consequence of this theorem, we can also state

Corollary 15.6. *If \mathcal{R} is a topologically filtered, topological relator on X , then $\text{res}_{\mathcal{R}}(U) \in \mathcal{N}_{\mathcal{R}}$ for all $U \in \mathcal{T}_{\mathcal{R}}$.*

Remark 15.7. Note that if \mathcal{R} is a topological relator on X and $U \in \mathcal{T}_{\mathcal{R}}$, then by Theorems 10.3, 10.2 and 9.3 we have $U = U^{\circ}$. Therefore, under the notation $U^{\ddagger} = \text{bnd}_{\mathcal{R}}(U)$, we also have $U^{\dagger} = U^{-} \setminus U = U^{-} \setminus U^{\circ} = U^{\ddagger}$.

Moreover, in Theorem 15.5 and Corollary 15.6, it is also enough to assume only that \mathcal{R} is a quasi-topologically filtered, topological relator on X . Namely, in this case, \mathcal{R} is already topologically filtered by Theorem 13.8.

16. A Few Basic Facts on Simple Relators

Definition 16.1. A relator \mathcal{R} on X to Y is called *properly simple* if it is a singleton relator. That is, there exists a relation R on X to Y such that $\mathcal{R} = \{R\}$.

Much more generally, the relator \mathcal{R} is called \mathfrak{F} -*simple*, for some structure \mathfrak{F} for relators on X to Y , if it is \mathfrak{F} -equivalent to a singleton relator. That is, there exists a relation R on X to Y such that $\mathfrak{F}\mathcal{R} = \mathfrak{F}R$.

Thus, by Theorems 6.4 and 8.12, for instance, we can at once state the following two theorems.

Theorem 16.2. For a relator \mathcal{R} on X on to Y , the following assertions are equivalent:

- (1) \mathcal{R} is \wedge -simple; (2) \mathcal{R} is cl -simple; (3) \mathcal{R} is int -simple.

Theorem 16.3. For a relator \mathcal{R} on X , the following assertions are equivalent:

- (1) \mathcal{R} is $\wedge\infty$ -simple; (2) \mathcal{R} is \mathcal{F} -simple; (3) \mathcal{R} is \mathcal{T} -simple.

Remark 16.4. Thus, for instance, the relator \mathcal{R} may be naturally called *topologically simple* if it is \wedge -simple.

Moreover, if in particular \mathcal{R} is a relator on X , then \mathcal{R} may be naturally called *quasi-topologically simple* if it is $\wedge\infty$ -simple.

Now, for instance, we can also easily prove the following

Theorem 16.5. If \mathcal{R} is a relator on X on to Y , then under the notation $R = \bigcap \mathcal{R}$ the following assertions are equivalent:

- (1) \mathcal{R} is topologically simple; (2) $R \in \mathcal{R}^\wedge$; (3) $\mathcal{R}^\wedge = \{R\}^\wedge$.

Proof. If (1) holds, then by the corresponding definitions there exists a relation S on X to Y such that $\mathcal{R}^\wedge = \{S\}^\wedge$. Hence, by using Corollary 4.10 and Theorem 6.4, we can infer that

$$R^{-1}(y) = \rho_{\mathcal{R}}(y) = \text{cl}_{\mathcal{R}}(\{y\}) = \text{cl}_S(\{y\}) = \rho_S(y) = S^{-1}(y)$$

for all $y \in Y$. Therefore, $R^{-1} = S^{-1}$, and thus also $R = S$. Consequently, we have $\mathcal{R}^\wedge = \{S\}^\wedge = \{R\}^\wedge$, and thus (3) also holds.

Now, since, (3) trivially implies (1), we need only show that (2) and (3) are also equivalent. For this, note that by Definition 6.1 we always have $R \in \{R\}^\wedge$ and $\mathcal{R} \subseteq \{R\}^\wedge$. Hence, by the corresponding properties of \wedge , it is clear that $\mathcal{R}^\wedge \subseteq \{R\}^\wedge$, and thus $\mathcal{R}^\wedge \subseteq \{R\}^\wedge$. Moreover, we can also note that

$$(2) \implies \{R\} \subseteq \mathcal{R}^\wedge \implies \{R\}^\wedge \subseteq \mathcal{R}^\wedge \implies \{R\}^\wedge \subseteq \mathcal{R}^\wedge.$$

Remark 16.6. Note that, by Definition 6.1, for any relation R on X to Y we actually have $\{R\}^\wedge = \{R\}^\# = \{R\}^*$. \square

While, for any relation S on X to Y , we have $S \in \{R\}^\wedge$ if and only if for each $x \in X$ there exists $\varphi(x) \in X$ such that $R(\varphi(x)) \subseteq S(x)$. That is, there exists a function φ of X to itself such that $R \circ \varphi \subseteq S$. Therefore, $\{R\}^\wedge = (R \circ X^X)^*$.

In addition to Theorem 16.5, we can also easily prove the following

Theorem 16.7. If \mathcal{R} is a nonvoid relator on X to Y , then under the notation $R = \bigcap \mathcal{R}$, we have

$$\mathcal{R}^{\vee\wedge} = \{R^{-1}\}^\wedge.$$

Proof. By Definitions 4.1 and 6.10 and Theorems 4.8, 7.1 and 4.9 and Corollary 4.10, for any $y \in Y$ and $A \subseteq X$, we have

$$\begin{aligned} y \in \text{cl}_{\mathcal{R}^\vee}(A) &\iff \{y\} \in \text{Cl}_{\mathcal{R}^\vee}(A) \iff \{y\} \in \text{Cl}_{\mathcal{R}^{\wedge-1}}(A) \iff \{y\} \in \text{Cl}_{\mathcal{R}^\wedge}^{-1}(A) \iff A \in \text{Cl}_{\mathcal{R}^\wedge}(\{y\}) \iff \\ &A \cap \text{cl}_{\mathcal{R}}(\{y\}) \neq \emptyset \iff A \cap \rho_{\mathcal{R}}(y) \neq \emptyset \iff A \cap R^{-1}(y) \neq \emptyset \iff y \in R[A] \iff y \in \text{cl}_{R^{-1}}(A). \end{aligned}$$

Therefore, $\text{cl}_{\mathcal{R}^\vee} = \text{cl}_{R^{-1}}$, and thus by Theorem 6.4 the required equality is true. \square

Now, as an immediate consequence of this theorem, we can also state

Corollary 16.8. For any nonvoid relator \mathcal{R} on X to Y , the relator \mathcal{R}^\vee is topologically simple.

Remark 16.9. Note that if in particular $\mathcal{R} = \emptyset$, but $X \neq \emptyset$, then by Definition 6.1 we have $\mathcal{R}^\wedge = \emptyset$, and thus also $\mathcal{R}^\vee = \mathcal{R}^{\wedge^{-1}} = \emptyset$ which cannot be topologically simple.

17. A Few Basic Facts on Symmetric Relators

In contrast to the reflexivity property of a relator \mathcal{R} on X , we may naturally introduce a great abundance of important symmetry and transitivity properties of the relator \mathcal{R} [148, 149].

For instance, the relator \mathcal{R} may be naturally called *strongly symmetric* if each member of \mathcal{R} is symmetric. Moreover, the relator \mathcal{R} may be naturally called *weakly symmetric* if the relation $R = \bigcap \mathcal{R}$ is symmetric.

However, it is now more important to note that we may also naturally have

Definition 17.1. A relator \mathcal{R} on X is called

- (1) *properly symmetric* if $R \in \mathcal{R}$ implies $R^{-1} \in \mathcal{R}$;
- (2) *uniformly symmetric* if for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S \subseteq R^{-1}$;
- (3) *proximally symmetric* if for each $A \subseteq X$ and $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S[A] \subseteq R^{-1}[A]$;
- (4) *topologically symmetric* if for each $x \in X$ and $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(x) \subseteq R^{-1}(x)$.

Remark 17.2. By using the operations -1 , $*$, $\#$ and \wedge , the above properties can be reformulated in more concise forms.

For instance, by using some basic properties of $\#$, we can easily prove

Theorem 17.3. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is proximally symmetric ;
- (2) $\mathcal{R}^{-1} \subseteq \mathcal{R}^\#$;
- (3) $\mathcal{R}^{-1\#} = \mathcal{R}^\#$;
- (4) $\mathcal{R}^{\#\#^{-1}} \subseteq \mathcal{R}^\#$;
- (5) $\mathcal{R}^{\#\#^{-1}} = \mathcal{R}^\#$.

Thus, in particular we can also state the following

Corollary 17.4. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) $\mathcal{R}^\#$ is properly symmetric ;
- (2) \mathcal{R} is proximally symmetric ;
- (3) \mathcal{R}^{-1} is proximally symmetric ;
- (4) \mathcal{R} and \mathcal{R}^{-1} are proximally equivalent .

Remark 17.5. Concerning topological symmetry, we can only prove that \mathcal{R} is topologically symmetric if and only if $\mathcal{R}^{-1} \subseteq \mathcal{R}^\wedge$, or equivalently $\mathcal{R}^{-1\wedge} \subseteq \mathcal{R}^\wedge$.

Hence, it is clear that, topological symmetry is already is not a genuine symmetry property. Therefore, in addition to Definition 17.1, we must also have

Definition 17.6. A relator \mathcal{R} on X is called

- (1) *properly topologically symmetric* if \mathcal{R}^\wedge is properly symmetric ;
- (2) *topologically bisymmetric* if both \mathcal{R} and \mathcal{R}^{-1} are topologically symmetric .

Remark 17.7. Thus, by Remark 17.5, we can see that \mathcal{R} is topologically bisymmetric if and only if $\mathcal{R}^\wedge = \mathcal{R}^{-1\wedge}$. That is, \mathcal{R} and \mathcal{R}^{-1} are topologically equivalent.

Moreover, by using Definitions 17.6 and 6.10 and Theorem 16.7, we can also easily prove the following two theorems.

Theorem 17.8. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is properly topologically symmetric ; (2) $\mathcal{R}^\vee \subseteq \mathcal{R}^\wedge$;
 (3) $\mathcal{R}^\vee = \mathcal{R}^\wedge$; (4) $\mathcal{R}^{\vee\wedge} = \mathcal{R}^\wedge$; (5) $\mathcal{R}^{\vee\vee} = \mathcal{R}^\vee$.

Theorem 17.9. If \mathcal{R} is a nonvoid relator on X , then under the notation $R = \bigcap \mathcal{R}$ the following assertions are equivalent :

- (1) \mathcal{R} is properly topologically symmetric ;
 (2) $R^{-1} \in \mathcal{R}^\wedge$; (3) $\mathcal{R}^\wedge = \{R^{-1}\}^\wedge$.

Proof. If (1) holds, then by Theorems 17.8 and 16.7 we have $\mathcal{R}^\wedge = \mathcal{R}^{\vee\wedge} = \{R^{-1}\}^\wedge$, and thus (3) holds.

While, if (3) holds, then

$$\mathcal{R}^{\wedge^{-1}} = \{R^{-1}\}^{\wedge^{-1}} = \{R^{-1}\}^{*-1} = \{R^{-1}\}^{-1*} = \{R\}^* = \{R\}^\wedge,$$

and thus (1) also holds. The equivalence of (2) and (3) is even more obvious. □

From this theorem, we can easily derive the following

Corollary 17.10. For a nonvoid relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is properly topologically symmetric ;
 (2) \mathcal{R} is topologically simple and weakly symmetric.

Proof. To derive the symmetry of $R = \bigcap \mathcal{R}$ from (1), note that if (1) holds, then by Corollary 4.10, Definition 4.1 and Theorems 17.9 and 6.4, we have

$$R^{-1} = \rho_{\mathcal{R}} = \rho_{\mathcal{R}^\wedge} = \rho_{\{R^{-1}\}^\wedge} = \rho_{R^{-1}} = R.$$

Remark 17.11. According to Remark 17.5, a relator \mathcal{R} on X may be naturally called *paratopologically symmetric* if $\mathcal{R}^{-1} \subseteq \mathcal{R}^\Delta$, or equivalently $\mathcal{R}^{-1\Delta} \subseteq \mathcal{R}^\Delta$. □

Moreover, by Definition 17.6, the relator \mathcal{R} may, for instance, be naturally called *properly paratopologically symmetric* if the relator \mathcal{R}^Δ is properly symmetric.

On the other hand, by Remark 17.5, the relator \mathcal{R} may also be naturally called *quasi-topologically symmetric* if $\mathcal{R}^{-1\wedge\infty} \subseteq \mathcal{R}^\wedge$, or equivalently $\mathcal{R}^{-1\wedge\infty} \subseteq \mathcal{R}^{\wedge\infty}$.

18. Some Further Theorems on Symmetric Relators

By Theorems 6.4 and 8.12, we can at once state the following two theorems.

Theorem 18.1. For a relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is topologically symmetric ;
 (2) $\text{cl}_{\mathcal{R}} \subseteq \text{cl}_{\mathcal{R}^{-1}}$; (3) $\text{int}_{\mathcal{R}^{-1}} \subseteq \text{int}_{\mathcal{R}}$.

Theorem 18.2. For a nonvoid relator \mathcal{R} on X , the following assertions are equivalent :

- (1) \mathcal{R} is quasi-topologically symmetric ;
 (2) $\mathcal{T}_{\mathcal{R}^{-1}} \subseteq \mathcal{T}_{\mathcal{R}}$; (3) $\mathcal{F}_{\mathcal{R}^{-1}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Remark 18.3. Thus, in particular we can also state \mathcal{R} is quasi-topologically bisymmetric if and only if $\mathcal{T}_{\mathcal{R}^{-1}} = \mathcal{T}_{\mathcal{R}}$, or equivalently $\mathcal{F}_{\mathcal{R}^{-1}} = \mathcal{F}_{\mathcal{R}}$.

However, it is now more important to note that, by using our former results, we can also prove the following three theorems.

Theorem 18.4. If \mathcal{R} is a nonvoid relator on X , then under the notation $R = \bigcap \mathcal{R}$ the following assertions are equivalent:

- (1) \mathcal{R} is properly topologically symmetric;
- (2) $\text{cl}_{\mathcal{R}}(A) = R[A]$ for all $A \subseteq X$;
- (3) $\text{int}_{\mathcal{R}}(A) = R[A^c]^c$ for all $A \subseteq X$.

Proof. To prove the implication (1) \implies (2), note that if (1) holds, then by Theorem 16.7 we have $\mathcal{R}^\wedge = \{R^{-1}\}^\wedge$. Hence, by using Theorems 6.4 and 4.9 we can infer that $\text{cl}_{\mathcal{R}}(A) = \text{cl}_{R^{-1}}(A) = R[A]$ for all $A \subseteq X$. Therefore, (2) also holds. \square

Theorem 18.5. For a nonvoid relator \mathcal{R} on X , the following assertions are equivalent:

- (1) \mathcal{R} is properly topologically symmetric;
- (2) $A \subseteq \text{int}_{\mathcal{R}}(B)$ implies $B^c \subseteq \text{int}_{\mathcal{R}}(A^c)$ for all $A, B \subseteq X$;
- (3) $A \cap \text{cl}_{\mathcal{R}}(B) \neq \emptyset$ implies $B \cap \text{cl}_{\mathcal{R}}(A) \neq \emptyset$ for all $A, B \subseteq X$.

Proof. To prove the implication (1) \implies (3), note that if (1) holds and $R = \bigcap \mathcal{R}$, then by Theorem 18.4 and Corollary 17.10

$$A \cap \text{cl}_{\mathcal{R}}(B) \neq \emptyset \implies A \cap R[B] \neq \emptyset \implies R^{-1}[A] \cap B \neq \emptyset \implies R[A] \cap B \neq \emptyset \implies \text{cl}_{\mathcal{R}}(A) \cap B \neq \emptyset$$

for all $A, B \subseteq X$. Therefore, (3) also holds. \square

Theorem 18.6. For a nonvoid relator \mathcal{R} on X , the following assertions are equivalent:

- (1) \mathcal{R} is properly topologically symmetric;
- (2) $A \subseteq \text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(A))$ for all $A \subseteq X$;
- (3) $\text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A)) \subseteq A$ for all $A \subseteq X$.

Proof. To prove the implication (1) \implies (2), note that if $A \subseteq X$, then by Theorem 4.5 we have $\text{cl}_{\mathcal{R}}(A)^c = \text{int}_{\mathcal{R}}(A^c)$, and thus also $\text{cl}_{\mathcal{R}}(A)^c \subseteq \text{int}_{\mathcal{R}}(A^c)$. Therefore, if (1) holds, then by Theorem 18.5 we also have $A \subseteq \text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(A))$. Thus, (2) also holds. \square

Remark 18.7. The latter three theorems can be naturally generalized to topologically simple relators.

For instance, it can be shown that a nonvoid relator on X to Y is topologically simple if and only if under the notation $\mathcal{S} = \mathcal{R}^{-1}$ or \mathcal{R}^\vee we have $A \subseteq \text{int}_{\mathcal{S}}(\text{cl}_{\mathcal{S}}(A))$ for all $A \subseteq X$.

19. Some Generalized Topologically Open Sets

Notation 19.1. In the sequel, we shall always assume that X is a set and \mathcal{R} is a relator on X .

Moreover, to shorten the subsequent proofs, we shall use the notations

$$A^- = \text{cl}_{\mathcal{R}}(A), \quad A^\circ = \text{int}_{\mathcal{R}}(A) \quad \text{and} \quad A^\dagger = \text{res}_{\mathcal{R}}(A).$$

Parts (2) and (3) of the following definition has been mainly suggested by [94, Theorem 1 and Definition 1] of Levine.

While, for the motivations of parts (1) and (4), see Corson and Michael [26], Mashhour et al. [102], and a survey by Dontchev [50].

Definition 19.2. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically*

- (1) *preopen* if $A \subseteq \text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(A))$;
- (2) *semi-open* if $A \subseteq \text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A))$;

- (3) *quasi-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq \text{cl}_{\mathcal{R}}(V)$;
 (4) *pseudo-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq \text{cl}_{\mathcal{R}}(A)$.

And, the families of all such subsets A of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = p, s, q$ and ps , respectively.

Remark 19.3. The inclusions $A \subseteq A^{-\circ}$ and $A \subseteq A^{\circ-}$ mean only that the set A is open with respect to the composite operations $-\circ$ and $\circ-$, respectively.

While, the inclusions $V \subseteq A \subseteq V^{-}$ and $A \subseteq V \subseteq A^{-}$ mean that A is near to V from above and below, or can be approximated by V from below and above.

The next simple example shows that, even in a very particular case, the families $\mathcal{T}_{\mathcal{R}}^q$ and $\mathcal{T}_{\mathcal{R}}^{ps}$ may be strictly smaller than the families $\mathcal{T}_{\mathcal{R}}^s$ and $\mathcal{T}_{\mathcal{R}}^p$, respectively.

Example 19.4. If $X = \{1, 2\}$ and R is a relation on X such that

$$R(1) = \{2\} \quad \text{and} \quad R(2) = \{1\}$$

then R is an injective function of X onto itself, with $R^{-1} = R$, such that

- (1) $\mathcal{T}_R = \mathcal{T}_R^q = \mathcal{T}_R^{ps} = \{\emptyset, X\}$; (2) $\mathcal{T}_R^s = \mathcal{T}_R^p = \mathcal{P}(X)$.

To check this, note that now, for any $A \subseteq X$, we have $R[A^c] = R[A]^c$. Therefore, by Theorems 4.9 and 4.5, we can state that

$$A^{-} = R^{-1}[A] = R[A] \quad \text{and} \quad A^{\circ} = A^{c-c} = R[A^c]^c = R[A]^{cc} = R[A].$$

Thus, by the corresponding definitions, we have $\mathcal{T}_R = \{\emptyset, X\}$ and $\mathcal{T}_R^q = \mathcal{T}_R^{ps} = \{\emptyset, X\}$. Moreover, for any $A \subseteq X$, we also have

$$A^{\circ-} = R[R[A]] = (R \circ R)[A] = (R \circ R^{-1})[A] = \Delta_X[A] = A,$$

and quite similarly $A^{-\circ} = A$. Therefore, $\mathcal{T}_R^s = \mathcal{T}_R^p = \mathcal{P}(X)$ also holds.

However, the appropriateness of Definition 19.2 can only be completely clear from the subsequent generalizations of the corresponding topological results.

Theorem 19.5. *We have*

- (1) $\mathcal{T}_{\mathcal{R}}^q \subseteq \mathcal{T}_{\mathcal{R}}^s$; (2) $\mathcal{T}_{\mathcal{R}}^{ps} \subseteq \mathcal{T}_{\mathcal{R}}^p$.

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^q$, then by Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that

$$V \subseteq A \subseteq V^{-}.$$

Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}$ and the increasingness of \circ , we can infer that $V \subseteq V^{\circ} \subseteq A^{\circ}$. Now, by using the increasingness of $-$, we can also see that

$$A \subseteq V^{-} \subseteq A^{\circ-}.$$

Therefore, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^s$.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{ps}$, then by Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that

$$A \subseteq V \subseteq A^{-}.$$

Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}$ and the increasingness of \circ , we can infer that

$$A \subseteq V \subseteq V^{\circ} \subseteq A^{-\circ}.$$

Therefore, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^p$. □

Theorem 19.6. *If \mathcal{R} is reflexive relator on X , then*

$$\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^q \cap \mathcal{T}_{\mathcal{R}}^{ps}.$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}}$, then by taking $V = A$ we have $V \in \mathcal{T}_{\mathcal{R}}$. And, by using Theorem 9.3, we can see that

$$V = A \subseteq A^- = V^- \quad \text{and} \quad A = V \subseteq V^- = A^-.$$

Therefore, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^q$ and $A \in \mathcal{T}_{\mathcal{R}}^{ps}$, and thus also $A \in \mathcal{T}_{\mathcal{R}}^q \cap \mathcal{T}_{\mathcal{R}}^{ps}$. \square

From this theorem, by using Theorem 19.5, we can immediately derive

Corollary 19.7. *If \mathcal{R} is reflexive relator on X , then $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p$.*

20. Further Basic Properties of Generalized Topologically Open Sets

In addition to Theorem 19.5, we can also prove the following

Theorem 20.1. *If \mathcal{R} is a topological relator on X , then*

$$(1) \mathcal{T}_{\mathcal{R}}^q = \mathcal{T}_{\mathcal{R}}^s; \quad (2) \mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^p.$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^s$, then by Definition 19.2, we have $A \subseteq A^{\circ-}$. Hence, by taking $V = A^{\circ}$, we get $A \subseteq V^-$. Moreover, by using Theorems 10.3, 9.3 and 10.2, we can see that

$$V = A^{\circ} \subseteq A \quad \text{and} \quad V = A^{\circ} \in \mathcal{T}_{\mathcal{R}}.$$

Thus, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^q$. This proves that $\mathcal{T}_{\mathcal{R}}^s \subseteq \mathcal{T}_{\mathcal{R}}^q$.

While, if $A \in \mathcal{T}_{\mathcal{R}}^p$, then by Definition 19.2 we have $A \subseteq A^{-\circ}$. Hence, by taking $V = A^{-\circ}$, we get $A \subseteq V$. Moreover, by using Theorems 10.3, 9.3 and 10.2, we can see that

$$V = A^{-\circ} \subseteq A^- \quad \text{and} \quad V = A^{-\circ} \in \mathcal{T}_{\mathcal{R}}.$$

Thus, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^{ps}$. This proves that $\mathcal{T}_{\mathcal{R}}^p \subseteq \mathcal{T}_{\mathcal{R}}^{ps}$.

Now, by Theorem 19.5, we can see that assertions (1) and (2) are also true. \square

By using our former results on topological relators, we can also prove

Theorem 20.2. *If \mathcal{R} is a topological relator on X , then for any $A \subseteq X$, the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^s$;
- (2) $\text{cl}_{\mathcal{R}}(A) = \text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A))$;
- (3) $\text{cl}_{\mathcal{R}}(A) = \text{cl}_{\mathcal{R}}(V)$ for some $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$.

Proof. If (1) holds, then by Definition 19.2 we have $A \subseteq A^{\circ-}$. Hence, by using the increasingness of $-$, we can infer that $A^- \subseteq A^{\circ--}$. Moreover, from Theorems 10.3 and 10.2, we can see that $A^{\circ-} \in \mathcal{F}_{\mathcal{R}}$. Thus, by the definition of $\mathcal{F}_{\mathcal{R}}$, we also have $A^{\circ--} \subseteq A^{\circ-}$. Therefore, $A^- \subseteq A^{\circ-}$.

On the other hand, by Theorems 10.3 and 9.3, we have $A^{\circ} \subseteq A$. Hence, by using the increasingness of $-$, we can infer that $A^{\circ-} \subseteq A^-$. Therefore, we actually have $A^- = A^{\circ-}$, and thus assertion (2) also holds.

While, if (2) holds, i. e., $A^- = A^{\circ-}$, then by taking $V = A^{\circ}$ we get $A^- = V^-$. Moreover, by Theorems 10.3, 9.3 and 10.2, we can see that $V \subseteq A$ and $V \in \mathcal{T}_{\mathcal{R}}$, and thus $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$. Therefore, (3) also holds.

Finally, if (3) holds, then there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A$ and $A^- = V^-$. Hence, by using Theorems 10.3 and 9.3, we can see that $A \subseteq A^- = V^-$. Therefore, by Definition 19.2, we have $A \in \mathcal{T}_{\mathcal{R}}^q$. Thus, by Theorem 19.5, assertion (1) also holds. \square

Remark 20.3. Note that if \mathcal{R} is a topological relator on X , then the family $\mathcal{T}_{\mathcal{R}}$ need not be closed under finite intersections.

Therefore, Theorem 20.2 is a generalization of the corresponding observations of Njåstad [110, p.961], Isomichi [75, Theorem 2], Noiri [111, Lemma 2] and Pipitone and Russo [123, Lemma 2.2].

Njåstad and Isomichi, being not aware of Levine's paper [94], investigated semi-open sets under the names " β -sets" and "subcondensed sets", respectively. While, Noiri, Pipitone and Russo already used Levine's terminology.

Theorem 20.2 is also a certain generalization of [6, Proposition 3.2] of Al-shami since, for every supra topology (generalized topology) \mathcal{T} on X , there exists a nonvoid preorder relator \mathcal{R} on X such that $\mathcal{T} = \mathcal{T}_{\mathcal{R}}$ [159].

Now, as an immediate consequence of Theorem 20.2, we can also state

Corollary 20.4. *If \mathcal{R} is a topological relator on X , then for any $A \subseteq X$, the following assertions are equivalent :*

- (1) $A \in \mathcal{F}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{R}}^s$;
- (2) $A = \text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A))$;
- (3) $A = \text{cl}_{\mathcal{R}}(V)$ for some $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$.

21. Another Important Property of Topologically Semi-Open and Quasi-Open Sets

By using similar arguments as in the proof of Theorem 20.2, we can also prove the following two generalizations of [94, Theorem 3] of Levine.

Theorem 21.1. *If \mathcal{R} is a quasi-topological relator on X ,*

$$A \in \mathcal{T}_{\mathcal{R}}^s \quad \text{and} \quad A \subseteq B \subseteq \text{cl}_{\mathcal{R}}(A),$$

then $B \in \mathcal{T}_{\mathcal{R}}^s$ also holds.

Proof. By Definition 19.2, we have $A \subseteq A^{\circ-}$. Hence, by using the increasingness of $-$, we can infer that $A^- \subseteq A^{\circ--}$. Moreover, by Theorem 10.2, we now also have $A^{\circ-} \in \mathcal{F}_{\mathcal{R}}$. Hence, by using the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $A^{\circ--} \subseteq A^{\circ-}$. Therefore, we also have $A^- \subseteq A^{\circ-}$.

On the other hand, because of the inclusion $A \subseteq B$ and the increasingness properties of \circ and $-$, we also have $A^{\circ-} \subseteq B^{\circ-}$. Therefore, $A^- \subseteq B^{\circ-}$ also holds. Hence, because of $B \subseteq A^-$, we can already see that $B \subseteq B^{\circ-}$. Thus, by Definition 19.2, the required assertion is also true. \square

Theorem 21.2. *If \mathcal{R} is a quasi-topological relator on X ,*

$$A \in \mathcal{T}_{\mathcal{R}}^q \quad \text{and} \quad A \subseteq B \subseteq \text{cl}_{\mathcal{R}}(A),$$

then $B \in \mathcal{T}_{\mathcal{R}}^q$ also holds.

Proof. By Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq V^-$. Hence, since $A \subseteq B$, it is clear that $V \subseteq B$. Moreover, by using the increasingness of $-$, we can also see that $A^- \subseteq V^{-}$.

On the other hand, by Theorem 10.2, we now also have $V^- \in \mathcal{F}_{\mathcal{R}}$. Hence, by the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $V^{-} \subseteq V^-$. Therefore, $A^- \subseteq V^-$ also holds. Hence, because of $B \subseteq A^-$, we can already see that $B \subseteq V^-$ also holds. Thus, by Definition 19.2, the required assertion is also true. \square

Remark 21.3. Note that, in the above proof, we have only used an apparently weaker property, than the quasi-topologicalness of \mathcal{R} .

Moreover, we have proved a little more than that was stated. Namely, that the “approximating set” V used for the set B does not depend on B .

Now, as an immediate consequence of Theorem 21.1, we can also state

Theorem 21.4. *If \mathcal{R} is a topological relator on X , then for every $A \in \mathcal{T}_{\mathcal{R}}^s$ we have $\text{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^s$.*

Proof. To derive this from Theorem 21.1, note that now by Theorems 10.3 the relator \mathcal{R} is reflexive and quasi-topological. Moreover, by Theorem 9.3, we have $A \subseteq A^-$, and thus also $A \subseteq A^- \subseteq A^-$. Therefore, by Theorem 21.1, we also have $A^- \in \mathcal{T}_{\mathcal{R}}^s$. \square

From this theorem, by using Corollary 19.7, we can immediately derive

Corollary 21.5. *If \mathcal{R} is a topological relator on X , then for every $V \in \mathcal{T}_{\mathcal{R}}$ we have $\text{cl}_{\mathcal{R}}(V) \in \mathcal{T}_{\mathcal{R}}^s$.*

Hence, by using Theorems 10.2 and 10.3, we can immediately derive

Corollary 21.6. *If \mathcal{R} is a topological relator on X , then for every $A \subseteq X$ we have $\text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A)) \in \mathcal{T}_{\mathcal{R}}^s$.*

Now, by using Theorems 19.6 and 21.1, we can also prove the following generalization [94, Theorem 5] of Levine.

Theorem 21.7. *If \mathcal{R} is a topological relator on X , then $\mathcal{A} = \mathcal{T}_{\mathcal{R}}^s$ is the smallest family of subsets of X such that*

- (1) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$; (2) $A \in \mathcal{A}$ and $A \subseteq B \subseteq \text{cl}_{\mathcal{R}}(A)$ imply $B \in \mathcal{A}$.

Proof. To prove the stated minimality property of $\mathcal{T}_{\mathcal{R}}^s$, note that if $A \in \mathcal{T}_{\mathcal{R}}^s$, then by Theorem 20.1 we also have $A \in \mathcal{T}_{\mathcal{R}}^q$. Thus, Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq V^-$. Therefore, if \mathcal{A} is a family of subsets of X such that (1) holds, then we have $V \in \mathcal{A}$. Moreover, if (2) also holds, then we also have $A \in \mathcal{A}$. Therefore, $\mathcal{T}_{\mathcal{R}}^s \subseteq \mathcal{A}$. \square

Moreover, we can also prove the following generalization of [94, Lemma 2] of Levine.

Theorem 21.8. *If \mathcal{R} is a topological relator on X , then*

$$\mathcal{T}_{\mathcal{R}} = \{\text{int}_{\mathcal{R}}(A) : A \in \mathcal{T}_{\mathcal{R}}^s\}.$$

Proof. Now, by Theorem 10.3, \mathcal{R} is reflexive and quasi-topological. Thus, if in particular $V \in \mathcal{T}_{\mathcal{R}}$, then by Corollary 19.7 we have also have $V \in \mathcal{T}_{\mathcal{R}}^s$.

Moreover, by the definition of $\mathcal{T}_{\mathcal{R}}$ and Theorem 9.3, we also have $V \subseteq V^\circ$ and $V^\circ \subseteq V$, and thus also $V = V^\circ$. Therefore, $V \in (\mathcal{T}_{\mathcal{R}}^s)^\circ$, and thus $\mathcal{T}_{\mathcal{R}} \subseteq (\mathcal{T}_{\mathcal{R}}^s)^\circ$ also holds.

On the other hand, by Theorem 10.2, we have $A^\circ \in \mathcal{T}_{\mathcal{R}}$ for all $A \subseteq X$. Thus, $\mathcal{P}(X)^\circ \subseteq \mathcal{T}_{\mathcal{R}}$, and thus in particular $(\mathcal{T}_{\mathcal{R}}^s)^\circ \subseteq \mathcal{T}_{\mathcal{R}}$ also holds. Therefore, the required equality $\mathcal{T}_{\mathcal{R}} = (\mathcal{T}_{\mathcal{R}}^s)^\circ$ is also true. \square

Remark 21.9. Note that, because of Theorem 20.1, here we may also write $\mathcal{T}_{\mathcal{R}}^q$ in place of $\mathcal{T}_{\mathcal{R}}^s$.

22. Another Important Property of Topologically Preopen and Pseudo-Open Sets

Analogously to Theorems 21.1 and 21.2, we can also prove the following two theorems.

Theorem 22.1. *If \mathcal{R} is a quasi-topological relator on X ,*

$$A \in \mathcal{T}_{\mathcal{R}}^p \text{ and } B \subseteq A \subseteq \text{cl}_{\mathcal{R}}(B),$$

then $B \in \mathcal{T}_{\mathcal{R}}^p$ also holds.

Proof. By Definition 19.2, we have $A \subseteq A^{-\circ}$. Hence, by using that $B \subseteq A$, we can see that $B \subseteq A^{-\circ}$.

Moreover, from the inclusion $A \subseteq B^-$, by using the increasingness of $-$ and Theorem 10.2, we can infer that

$$A^- \subseteq B^{-\circ} \subseteq B^-.$$

Hence, by using the increasingness of \circ , we can infer that $A^{-\circ} \subseteq B^{-\circ}$. Therefore, because of $B \subseteq A^{-\circ}$, we also have $B \subseteq B^{-\circ}$. Hence, by Definition 19.2, we can see that $B \in \mathcal{T}_{\mathcal{R}}^p$ also holds. \square

Theorem 22.2. *If \mathcal{R} is a quasi-topological relator on X ,*

$$A \in \mathcal{T}_{\mathcal{R}}^{sp} \text{ and } B \subseteq A \subseteq \text{cl}_{\mathcal{R}}(B),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{sp}$ also holds.

Proof. By Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^-$. Hence, by using that $B \subseteq A$, we can see that $B \subseteq V$.

Moreover, from the inclusion $A \subseteq B^-$, by using the increasingness of $-$ and Theorem 10.2, we can infer that

$$A^- \subseteq B^{-\circ} \subseteq B^-.$$

Hence, by using the inclusion $V \subseteq A^-$, we can see that $V \subseteq B^-$. Therefore, we actually have $B \subseteq V \subseteq B^-$. Hence, by Definition 19.2, we can see that $B \in \mathcal{T}_{\mathcal{R}}^{sp}$ also holds. \square

Remark 22.3. Again, we have proved a little more than that was stated. Namely, that the “approximating set” V used for the set B does not depend on B .

Instead of some counterparts of Theorem 21.4 and its corollaries, we shall only prove here the following two theorems.

Theorem 22.4. *If \mathcal{R} is a nonvoid relator on X , then $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^p$.*

Proof. If $A \in \mathcal{D}_{\mathcal{R}}$, then by the definition of $\mathcal{D}_{\mathcal{R}}$ we have $A^- = X$. Moreover, since $\mathcal{R} \neq \emptyset$, by the definition of \circ , we also have $X^\circ = X$. Therefore, we actually have $A^{-\circ} = X^\circ = X$. Thus, the required inclusion $A \subseteq A^{-\circ}$ trivially holds. \square

Theorem 22.5. *If \mathcal{R} is a topologically filtered relator on X and $A = V \cap B$ for some $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{D}_{\mathcal{R}}$, then $A \in \mathcal{T}_{\mathcal{R}}^{ps}$.*

Proof. In this case, we have $A \subseteq V$. Moreover, by the definition of $\mathcal{D}_{\mathcal{R}}$ and Corollary 14.4, we also have

$$V = X \cap V = B^- \cap V \subseteq (B \cap V)^- = A^-.$$

Therefore, we actually have $A \subseteq V \subseteq A^-$. Thus, by Definition 19.2, we have $A \in \mathcal{T}_{\mathcal{R}}^{ps}$. \square

From this theorem, by taking $V = X$ whenever $\mathcal{R} \neq \emptyset$, we can get

Corollary 22.6. *If \mathcal{R} is a nonvoid, topologically filtered relator on X , then $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{ps}$.*

Now, as some close analogues of Theorems 21.7 and 21.8, we can also prove the following two theorems.

Theorem 22.7. *If \mathcal{R} is a topological relator on X , then $\mathcal{A} = \mathcal{T}_{\mathcal{R}}^p$ is the smallest family of subsets of X such that*

- (1) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$; (2) $A \in \mathcal{A}$ and $B \subseteq A \subseteq \text{cl}_{\mathcal{R}}(B)$ imply $B \in \mathcal{A}$.

Proof. To prove the stated minimality property of $\mathcal{T}_{\mathcal{R}}^p$, note that if $A \in \mathcal{T}_{\mathcal{R}}^p$, then by Theorem 20.1 we also have $A \in \mathcal{T}_{\mathcal{R}}^{ps}$. Thus, Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^-$. Therefore, if \mathcal{A} is a family of subsets of X such that (1) holds, then we have $V \in \mathcal{A}$. Moreover, if (2) also holds, then we also have $A \in \mathcal{A}$. Therefore, $\mathcal{T}_{\mathcal{R}}^p \subseteq \mathcal{A}$. \square

Theorem 22.8. *If \mathcal{R} is a topological relator on X , then*

$$\mathcal{T}_{\mathcal{R}} = \{ \text{int}_{\mathcal{R}}(A) : A \in \mathcal{T}_{\mathcal{R}}^p \}.$$

Proof. Now, by Theorem 10.3, \mathcal{R} is reflexive and quasi-topological. Thus, if in particular $V \in \mathcal{T}_{\mathcal{R}}$, then by Corollary 19.7 we have also have $V \in \mathcal{T}_{\mathcal{R}}^p$.

Moreover, by the definition of $\mathcal{T}_{\mathcal{R}}$ and Theorem 9.3, we also have $V \subseteq V^\circ$ and $V^\circ \subseteq V$, and thus also $V = V^\circ$. Therefore, $V \in (\mathcal{T}_{\mathcal{R}}^p)^\circ$, and thus $\mathcal{T}_{\mathcal{R}} \subseteq (\mathcal{T}_{\mathcal{R}}^p)^\circ$ also holds.

On the other hand, by Theorem 10.2, we have $A^\circ \in \mathcal{T}_{\mathcal{R}}$ for all $A \subseteq X$. Thus, $\mathcal{P}(X)^\circ \subseteq \mathcal{T}_{\mathcal{R}}$, and thus in particular $(\mathcal{T}_{\mathcal{R}}^p)^\circ \subseteq \mathcal{T}_{\mathcal{R}}$ also holds. Therefore, the required equality $\mathcal{T}_{\mathcal{R}} = (\mathcal{T}_{\mathcal{R}}^p)^\circ$ is also true. \square

Remark 22.9. Note that, because of Theorem 20.1, here we may also write $\mathcal{T}_{\mathcal{R}}^{ps}$ in place of $\mathcal{T}_{\mathcal{R}}^p$.

23. Topologically Regular Open Sets

By Kuratowski [87], in addition to Definition 19.2, we may also naturally introduce the following

Definition 23.1. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically regular open* if

$$A = \text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(A)).$$

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^r$.

Remark 23.2. Note that thus \mathcal{T}_R^r is just the family of all fixed points of the composite operation $- \circ = -c - c$.

However, \mathcal{T}_R^r need not be a subset of \mathcal{T}_R . Namely, if X and R are as in Example 19.4, then $\mathcal{T}_R = \{\emptyset, X\}$, but $\mathcal{T}_R^r = \mathcal{P}(X)$.

Of course, as an immediate consequence of Theorem 10.2, we can state

Theorem 23.3. *If \mathcal{R} is a quasi-topological relator on X , then $\mathcal{T}_R^r \subseteq \mathcal{T}_R$.*

Moreover, by using the plausible notation $\mathcal{F}_R^s = \{A^c : A \in \mathcal{T}_R^s\}$, we can prove the following generalization of a statement of Dontchev [50, p. 4]. (See also Ekici [54, Theorem 8] and Jamunarani et al. [77, Theorem 2.2] for some more general results.)

Theorem 23.4. *We have*

$$\mathcal{T}_R^r = \mathcal{T}_R^p \cap \mathcal{F}_R^s.$$

Proof. By Definitions 23.1 and 19.2, it is clear that, for any $A \subseteq X$, we have

$$A \in \mathcal{T}_R^r \iff A = A^{-\circ} \iff A \subseteq A^{-\circ}, \quad A^{-\circ} \subseteq A \iff A \in \mathcal{T}_R^p, \quad A^c \subseteq A^{-\circ c}.$$

Moreover, by using the equalities $\circ c = c -$ and $-c = c \circ$, we can see that

$$A^{-\circ c} = A^{-c-} = A^{c^{\circ-}}.$$

Therefore,

$$A^c \subseteq A^{-\circ c} \iff A^c \subseteq A^{c^{\circ-}} \iff A^c \in \mathcal{T}_R^s \iff A \in \mathcal{F}_R^s.$$

Consequently,

$$A \in \mathcal{T}_R^r \iff A \in \mathcal{T}_R^p, \quad A \in \mathcal{F}_R^s \iff A \in \mathcal{T}_R^p \cap \mathcal{F}_R^s,$$

and thus the required equality is true. □

From this theorem, by using Theorem 20.1, we can immediately derive

Corollary 23.5. *If \mathcal{R} is a topological relator on X , then $\mathcal{T}_R^r = \mathcal{T}_R^{ps} \cap \mathcal{F}_R^q$.*

Proof. To check this, note that by Theorem 20.1, for any $A \subseteq X$, we have

$$A \in \mathcal{F}_R^s \iff A^c \in \mathcal{T}_R^s \iff A^c \in \mathcal{T}_R^q \iff A \in \mathcal{F}_R^q.$$

Moreover, from Theorem 23.4, by using Corollary 19.7 and Theorem 23.3, we can easily derive the following partial dual of Corollary 20.4. □

Theorem 23.6. *If \mathcal{R} is a topological relator on X , then*

$$\mathcal{T}_R^r = \mathcal{T}_R \cap \mathcal{F}_R^s.$$

Proof. By Theorem 10.3, \mathcal{R} is reflexive and quasi-topological. Thus, by Corollary 19.7 and Theorem 23.3, we have $\mathcal{T}_R \subseteq \mathcal{T}_R^p$ and $\mathcal{T}_R^r \subseteq \mathcal{T}_R$. Hence, by using Theorem 23.4, we can already infer that

$$\mathcal{T}_R^r = \mathcal{T}_R \cap \mathcal{T}_R^r = \mathcal{T}_R \cap \mathcal{T}_R^p \cap \mathcal{F}_R^s = \mathcal{T}_R \cap \mathcal{F}_R^s.$$

From this theorem, by using Theorem 20.1, we can immediately derive □

Corollary 23.7. *If \mathcal{R} is a topological relator on X , then $\mathcal{T}_R^r = \mathcal{T}_R \cap \mathcal{F}_R^q$.*

By using Theorem 23.6, we can also easily prove the following generalization of a statement of Kuratowski [87].

Theorem 23.8. *If \mathcal{R} is a topological relator on X , then for any $A \in \mathcal{T}_{\mathcal{R}}^s$ we have*

$$\text{cl}_{\mathcal{R}}(A)^c \in \mathcal{T}_{\mathcal{R}}^r.$$

Proof. By Theorem 23.6, we have $\mathcal{T}_{\mathcal{R}}^r = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^s$. Moreover, by Theorems 10.3 and 10.2, we have $A^- \in \mathcal{F}_{\mathcal{R}}$, and thus $A^{-c} \in \mathcal{T}_{\mathcal{R}}$. Therefore, to obtain the required assertion $A^{-c} \in \mathcal{T}_{\mathcal{R}}^r$, we need only show that $A^{-c} \in \mathcal{F}_{\mathcal{R}}^s$, i. e., $A^- \in \mathcal{T}_{\mathcal{R}}^s$ also holds. However, this is true by Theorem 21.4. \square

Remark 23.9. Note that thus, under the conditions of the above theorem, we also have $A^{-\circ} = A^{-c-c} \in \mathcal{T}_{\mathcal{R}}^r$.

However, the latter fact seems to be of no importance, since by using Theorem 23.6 we can also prove the following

Theorem 23.10. *If \mathcal{R} is a topological relator on X , then for any $A \subseteq X$ we have*

$$\text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(A)) \in \mathcal{T}_{\mathcal{R}}^r.$$

Proof. By Theorem 23.6, we have $\mathcal{T}_{\mathcal{R}}^r = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^s$. Moreover, by Theorems 10.3 and 10.2, we have $A^{-\circ} \in \mathcal{T}_{\mathcal{R}}$. Therefore, to obtain the required assertion $A^{-\circ} \in \mathcal{T}_{\mathcal{R}}^r$, we need only show that $A^{-\circ} \in \mathcal{F}_{\mathcal{R}}^s$, i. e., $A^{-\circ c} \in \mathcal{T}_{\mathcal{R}}^s$ also holds.

For this, note that, because of the equalities $\circ c = c -$ and $-c = c \circ$ and Corollary 21.6, we have

$$A^{-\circ c} = A^{-c-} = A^{c\circ-} \in \mathcal{T}_{\mathcal{R}}^s.$$

Remark 23.11. The above two theorems can also be proved directly, by using only Theorems 10.3, 9.3 and 10.2 and the increasingness of the operations $-$ and \circ . \square

By using Theorems 12.5, 23.8 and 23.10, it can be shown that if \mathcal{R} is a topologically filtered, topological relator on X , then the family $\mathcal{T}_{\mathcal{R}}^r$ forms a Boolean algebra under the operations defined by

$$A' = A^{-c}, \quad A \wedge B = A \cap B \quad \text{and} \quad A \vee B = (A \cup B)''$$

for all $A, B \in \mathcal{T}_{\mathcal{R}}^r$. (See Stone [144] and Givant and Halmos [64].)

However, the above statement is of no particular importance for us now, since unions, intersections and complements of generalized topologically open sets will only be investigated in a subsequent paper.

24. Characterizations of topologically semi-open and quasi-open sets

Now, as a generalization of [51, Lemma 1] of Duszyński and Noiri, we can also prove the following

Theorem 24.1. *If \mathcal{R} is a reflexive relator on X , then for any $A \subseteq X$ the following assertions are equivalent:*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^s$;
- (2) there exists $B \subseteq X$ such that

$$A = \text{int}_{\mathcal{R}}(A) \cup B \quad \text{and} \quad B \subseteq \text{res}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A)).$$

Proof. By Theorem 9.3, we have $A^{\circ} \subseteq A$. Moreover, if (1) holds, then by Definition 19.2 we have $A \subseteq A^{\circ-}$. Hence, by defining $B = A \setminus A^{\circ}$, we can already see that

$$A = A^{\circ} \cup (A \setminus A^{\circ}) = A^{\circ} \cup B \quad \text{and} \quad B = A \setminus A^{\circ} \subseteq A^{\circ-} \setminus A^{\circ} = A^{\circ\ddagger}.$$

Therefore, (2) also holds.

By Theorem 9.3, we also have $A^{\circ} \subseteq A^{\circ-}$. Therefore, if (2) holds, then we have

$$A = A^{\circ} \cup B \subseteq A^{\circ} \cup A^{\circ\ddagger} = A^{\circ} \cup (A^{\circ-} \setminus A^{\circ}) = A^{\circ-}.$$

Thus, by Definition 19.2, assertion (1) also holds. \square

Remark 24.2. Note that if B is as in (2), then $A = A^\circ \cup B$. And, because of $B \subseteq A^{\circ\ddagger} = A^{\circ-} \setminus A^\circ$, we have $A^\circ \cap B = \emptyset$.

Moreover, by Theorem 9.3, we have $A^{\circ\circ} \subseteq A^\circ$. Therefore, by using the notation $A^\ddagger = \text{bnd}_{\mathcal{R}}(A)$, we can also state that $B \subseteq A^{\circ\ddagger} = A^{\circ-} \setminus A^\circ \subseteq A^{\circ-} \setminus A^{\circ\circ} = A^{\circ\ddagger}$.

Now, by using our former results, we can also easily prove the following generalization of an observation of Dłaska, Ergun and Ganster [47, p. 1163].

Theorem 24.3. *If \mathcal{R} is a topological relator on X , then for any $A \subseteq X$ the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^s$;
- (2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \subseteq X$ such that

$$A = V \cup B \quad \text{and} \quad B \subseteq \text{res}_{\mathcal{R}}(V).$$

Proof. If (1) holds, then by Theorem 24.1, there exists $B \subseteq X$ such that

$$A = A^\circ \cup B \quad \text{and} \quad B \subseteq A^{\circ\ddagger}.$$

Moreover, by Theorems 10.3 and 10.2, we have $A^\circ \in \mathcal{T}_{\mathcal{R}}$. Hence, by taking $V = A^\circ$, we can see that (2) also holds.

On the other hand, by Theorems 10.3 and 9.3, we have $V \subseteq V^-$ for any $V \subseteq X$. Therefore, if (2) holds, then we have not only $V \subseteq A$ but also

$$A = V \cup B \subseteq V \cup V^\ddagger = V \cup (V^- \setminus V) = V^-.$$

Hence, by Definition 19.2, we can see that $A \in \mathcal{T}_{\mathcal{R}}^q$. Thus, by Theorem 20.1, assertion (1) also holds. \square

Remark 24.4. Note that if in particular \mathcal{R} is a topologically filtered, topological relator on X , then by Corollary 15.6 we have $V^\ddagger \in \mathcal{N}_{\mathcal{R}}$. Hence, since $B \subseteq V^\ddagger$, it is clear that $B \in \mathcal{N}_{\mathcal{R}}$ also holds.

Therefore, analogously to [94, Theorem 7] of Levine, we can also state

Theorem 24.5. *If \mathcal{R} is a topologically filtered, topological relator on X and $A \in \mathcal{T}_{\mathcal{R}}^s$, then there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that*

$$A = V \cup B \quad \text{and} \quad V \cap B = \emptyset.$$

The following obvious reformulation of [94, Example 2] of Levine shows that the converse of this theorem is false.

Example 24.6. Define $X = \mathbb{R}$ and

$$R_n = \{(x, y) \in X^2 : d(x, y) < n^{-1}\}$$

for all $n \in \mathbb{N}$.

Then, $\mathcal{R} = \{R_n : n \in \mathbb{N}\}$ is a properly filtered, strongly topological relator on X such that, under the notations

$$V =]0, 1[\quad \text{and} \quad B = \{2\},$$

we have $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \cup B \notin \mathcal{T}_{\mathcal{R}}^s$.

To check the required properties, note that

- (1) $R_n = R_n \cap R_m$ if $n, m \in \mathbb{N}$ such that $m \leq n$;
- (2) $R_n(x) =]x - n^{-1}, x + n^{-1}[$ for all $n \in \mathbb{N}$ and $x \in X$;
- (3) for each $n \in \mathbb{N}$, $x \in X$ and $y \in R_n(x)$, we have $R_m(y) \subseteq R_n(x)$ if $m \in \mathbb{N}$ such that $m^{-1} < n^{-1} - d(x, y)$.

By using (2), we can easily see that $A^\circ = V$ and $A^{\circ-} = V^- = [0, 1]$, and thus $A \notin \mathcal{T}_{\mathcal{R}}^s$. Therefore, by Definition 19.2, $A \notin \mathcal{T}_{\mathcal{R}}^s$. Although, $A = V \cup B$, $V \cap B = \emptyset$, $V \in \mathcal{T}_{\mathcal{R}}$ and $B^{\circ-} = B^\circ = \emptyset$.

While, by using assertion (3) and the property $\sup(\mathbb{N}) = +\infty$, we can easily see that $R_n(x) \in \mathcal{T}_{\mathcal{R}}$ for all $n \in \mathbb{N}$ and $x \in X$. Therefore, \mathcal{R} is strongly quasi-topological.

Remark 24.7. Assertion (3) can also be easily derived from (2). However, it seems not to be a consequence of the very strong reflexivity, symmetry and transitivity properties of \mathcal{R} .

Now, in addition to $\Delta_X \subseteq R_n$ and $R_n = R_n^{-1}$, we also have $R_{2n} \circ R_{2n} \subseteq R_n$ for all $n \in \mathbb{N}$. Thus, \mathcal{R} is a strictly uniformly transitive tolerance relator on X .

25. Characterizations of preopen and pseudo-open sets

As a counterpart of [59, Proposition 2.1] of Ganster, we can also prove

Theorem 25.1. *If \mathcal{R} is a topologically filtered, topological relator on X , then for any $A \subseteq X$ the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^p$;
- (2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$;
- (3) there exist $V \in \mathcal{T}_{\mathcal{R}}^r$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$.

Proof. If (1) holds, then by Definition 19.2 we have $A \subseteq A^{-\circ}$. Hence, we can infer that

$$A = A^{-\circ} \setminus (A^{-\circ} \setminus A) = A^{-\circ} \cap (A^{-\circ} \cap A^c)^c = A^{-\circ} \cap (A^{-\circ c} \cup A).$$

Now, by defining

$$V = A^{-\circ} \quad \text{and} \quad B = A^{-\circ c} \cup A,$$

we can state that $A = V \cap B$. And, by Theorem 23.10, we can state that $V = A^{-\circ} \in \mathcal{T}_{\mathcal{R}}^r$.

Moreover, by using Theorem 12.5, we can see that

$$B^- = (A^{-\circ c} \cup A)^- = A^{-\circ c^-} \cup A^-.$$

And, by using the equality $\circ c = c^-$ and Theorems 10.3 and 9.3, we can also see that

$$A^{-\circ c^-} = A^{-c^-} \supseteq A^{-c} \supseteq A^-.$$

Therefore,

$$B^- = A^{-\circ c^-} \cup A^- \supseteq A^{-c} \cup A^- = X,$$

and thus $B^- = X$. Hence, by the definition of $\mathcal{D}_{\mathcal{R}}$, we can see that $B \in \mathcal{D}_{\mathcal{R}}$, and thus (3) also holds.

From Theorem 23.6, we know that $\mathcal{T}_{\mathcal{R}}^r \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, assertion (2) is an immediate consequence of (3). Moreover, from Theorems 22.5 and 19.5, we can see that (2) implies (1) even if the relator \mathcal{R} is assumed to be only topologically filtered. \square

In addition to the above theorem, we can also prove the following

Theorem 25.2. *If \mathcal{R} is a topologically filtered, topological relator on X , then for any $A \subseteq X$ the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^p$;
- (2) there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V$ and $\text{cl}_{\mathcal{R}}(A) = \text{cl}_{\mathcal{R}}(V)$;
- (3) there exists $V \in \mathcal{T}_{\mathcal{R}}^r$ such that $A \subseteq V$ and $\text{cl}_{\mathcal{R}}(A) = \text{cl}_{\mathcal{R}}(V)$.

Proof. If (1) holds, then by Theorem 25.1 there exist $V \in \mathcal{T}_{\mathcal{R}}^r$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$. Thus, in particular $A \subseteq V$. Hence, by using the increasingness $-$, we can infer that $A^- \subseteq V^-$.

Moreover, by Theorem 23.6, we have $\mathcal{T}_{\mathcal{R}}^r \subseteq \mathcal{T}_{\mathcal{R}}$, and thus $V \in \mathcal{T}_{\mathcal{R}}$. Therefore, by the definition of $\mathcal{D}_{\mathcal{R}}$ and Corollary 14.4, we can also state that

$$V = V \cap X = V \cap B^- \subseteq (V \cap B)^- = A^-.$$

Hence, by using the increasingness of $-$, Theorems 10.3 and 10.2 and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $V^- \subseteq A^{--} \subseteq A^-$. Therefore, we actually have $A^- = V^-$, and thus (3) also holds.

Again, by the inclusion $\mathcal{T}_{\mathcal{R}}^s \subseteq \mathcal{T}_{\mathcal{R}}$, it is clear that (2) is an immediate consequence of (3). Therefore, we need only show that (2) also implies (1).

For this, note that if (2) holds, then by Theorems 10.3 and 9.3, in addition to $A \subseteq V$, we also have $V \subseteq V^- = A^-$. Therefore, by Definition 19.2, we have $A \in \mathcal{T}_{\mathcal{R}}^{ps}$. Hence, by Theorem 19.5, we can see that (1) also holds. \square

Remark 25.3. Note that, by Theorem 20.1, in the above two theorems we may again write $\mathcal{T}_{\mathcal{R}}^{ps}$ in place of $\mathcal{T}_{\mathcal{R}}^p$.

26. A further important property of topologically semi-open and preopen sets

By using Corollary 14.4, we can also prove the following improvement of [112, Lemma 2.5] of Noiri.

Theorem 26.1. *If \mathcal{R} is a topologically filtered, topological relator on X , and*

$$A \in \mathcal{T}_{\mathcal{R}}^s, \quad B \in \mathcal{T}_{\mathcal{R}}^p, \quad \text{int}_{\mathcal{R}}(A) \subseteq C \subseteq \text{cl}_{\mathcal{R}}(A), \quad B \subseteq D \subseteq X,$$

then

$$\text{cl}_{\mathcal{R}}(A) \cap B = \text{cl}_{\mathcal{R}}(C \cap D) \cap B.$$

Proof. By Definition 19.2, we have

$$A \subseteq A^{\circ-} \quad \text{and} \quad B \subseteq B^{-\circ}.$$

Hence, by using the increasingness of the operation $-$, Theorems 10.3 and 10.2, and the definition $\mathcal{F}_{\mathcal{R}}$, we can infer that

$$A^- \subseteq A^{\circ--} \subseteq A^{\circ-}.$$

Moreover, from Theorems 10.3, 10.2 and 9.3, we can see that $A^{\circ}, B^{-\circ} \in \mathcal{T}_{\mathcal{R}}$ and $B^{-\circ} \subseteq B^-$. Hence, by using Corollary 14.4, the corresponding properties of $-$, and the assumptions $A^{\circ} \subseteq C$ and $B \subseteq D$, we can infer that

$$A^- \cap B \subseteq A^{\circ-} \cap B^{-\circ} \subseteq (A^{\circ} \cap B^{-\circ})^- \subseteq (A^{\circ} \cap B^-)^- \subseteq (A^{\circ} \cap B)^{-} \subseteq (A^{\circ} \cap B)^- \subseteq (C \cap D)^-,$$

and thus $A^- \cap B \subseteq (C \cap D)^- \cap B$.

Moreover, by using the corresponding properties of $-$ and the assumption $C \subseteq A^-$, we can also see that

$$(C \cap D)^- \subseteq C^- \subseteq A^{--} \subseteq A^-,$$

and thus $(C \cap D)^- \cap B \subseteq A^- \cap B$. Therefore, the required equality is also true. \square

From this theorem, by choosing C and D appropriately, we can immediately derive a great number equalities for the set $A^- \cap B$. For instance, we can at once state the following

Corollary 26.2. *If \mathcal{R} is a topologically filtered, topological relator on X , and $A \in \mathcal{T}_{\mathcal{R}}^s$ and $B \in \mathcal{T}_{\mathcal{R}}^p$, then*

$$\text{cl}_{\mathcal{R}}(A) \cap B = \text{cl}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A)) \cap \text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(B))) \cap B.$$

Theorem 26.1 strongly suggests that some of the equalities stated by Dontchev [50, p. 4], without proofs and references, cannot be true.

To see that this is really the case, we can use the following

Example 26.3. If X and \mathcal{R} are as in Example 24.6, and

$$A = [0, 1] \quad \text{and} \quad B =]0, 1[\cap \mathbb{Q},$$

then \mathcal{R} is a properly filtered, strongly topological relator on X , and $A \in \mathcal{T}_{\mathcal{R}}^s$ and $B \in \mathcal{T}_{\mathcal{R}}^p$ such that

$$\text{cl}_{\mathcal{R}}(A) \cap B =]0, 1[\cap \mathbb{Q} \quad \text{and} \quad \text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A) \cap B) = [0, 1],$$

and thus even $\text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A) \cap B) \not\subseteq \text{cl}_{\mathcal{R}}(A) \cap B$.

Note that the inclusions $A \in \mathcal{T}_{\mathcal{R}}^s$ and $B \in \mathcal{T}_{\mathcal{R}}^p$ are immediate consequences of Corollary 21.5 and Theorems 22.5 and 19.5.

27. Some further generalized topologically open sets

Parts (1) and (2) of the following definition has been suggested by Njåstad [110] and Abd El-Monsef et al. [1]. While, for some motivations of parts (3) and (4), see Theorems 21.1 and 22.1 and [8, 2.1 Definition] of Andrijević.

Definition 27.1. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically*

- (1) α -open if $A \subseteq \text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A)))$;
- (2) β -open if $A \subseteq \text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(A)))$;
- (3) γ -open if there exists $V \in \mathcal{T}_{\mathcal{R}}^s$ such that $A \subseteq V \subseteq \text{cl}_{\mathcal{R}}(A)$;
- (4) δ -open if there exists $V \in \mathcal{T}_{\mathcal{R}}^p$ such that $V \subseteq A \subseteq \text{cl}_{\mathcal{R}}(V)$.

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = \alpha, \beta, \gamma$ and δ , respectively.

Remark 27.2. Note that if the relator \mathcal{R} is not topological, then by using the families $\mathcal{T}_{\mathcal{R}}^q$ and $\mathcal{T}_{\mathcal{R}}^{ps}$ instead of $\mathcal{T}_{\mathcal{R}}^s$ and $\mathcal{T}_{\mathcal{R}}^p$, respectively, we can get some stronger forms of generalized topologically open sets.

Now, by using Definition 27.1 and the increasingness of \circ and $-$, we can easily prove the following

Theorem 27.3. We have

$$\mathcal{T}_{\mathcal{R}}^{\gamma} \cup \mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}.$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$, then by Definition 27.1 there exists $V \in \mathcal{T}_{\mathcal{R}}^s$ such that

$$A \subseteq V \subseteq A^{-}.$$

Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}^s$ and the increasingness of \circ and $-$, we can see that

$$A \subseteq V \subseteq V^{\circ-} \subseteq A^{-\circ-}.$$

Therefore, by Definition 27.1, we also have $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{\delta}$, then by Definition 27.1 there exists $V \in \mathcal{T}_{\mathcal{R}}^p$ such that

$$V \subseteq A \subseteq V^{-}.$$

Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}^p$ and the increasingness of $-$ and $- \circ -$, we can see that

$$A \subseteq V^{-} \subseteq V^{-\circ-} \subseteq A^{-\circ-}.$$

Therefore, by Definition 27.1, we also have $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$. □

Moreover, by Definition 27.1, Theorem 9.3 and Corollary 19.7, it is clear that we also have the following

Theorem 27.4. If \mathcal{R} is a reflexive relator on X , then

- (1) $\mathcal{T}_{\mathcal{R}}^s \cup \mathcal{T}_{\mathcal{R}}^{ps} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma}$;
- (2) $\mathcal{T}_{\mathcal{R}}^p \cup \mathcal{T}_{\mathcal{R}}^q \subseteq \mathcal{T}_{\mathcal{R}}^{\delta}$.

Proof. To prove (1), note that if $A \in \mathcal{T}_{\mathcal{R}}^s$, then by taking $V = A$ we evidently have $V \in \mathcal{T}_{\mathcal{R}}^s$ such that $A \subseteq V$. Moreover, by Theorem 9.3, we can also see that $V \subseteq V^{-} = A^{-}$. Thus, by Definition 27.1, $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ also holds.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{ps}$, then by Definition 19.2 there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^{-}$. Moreover, by Corollary 19.7, we have $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^s$, and thus in particular $V \in \mathcal{T}_{\mathcal{R}}^s$. Therefore, by Definition 27.1, $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ also holds. □

Now, in addition to Theorem 19.6 and Corollary 19.7, we can also prove

Theorem 27.5. If \mathcal{R} is a reflexive relator on X , then $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\kappa}$ also holds with $\kappa = \alpha, \beta, \gamma$ and δ .

Proof. By Corollary 19.7 and Theorems 27.4 and 27.3, we have

$$\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p \subseteq \mathcal{T}_{\mathcal{R}}^\gamma \cap \mathcal{T}_{\mathcal{R}}^\delta \subseteq \mathcal{T}_{\mathcal{R}}^\gamma \cup \mathcal{T}_{\mathcal{R}}^\delta \subseteq \mathcal{T}_{\mathcal{R}}^\beta.$$

Therefore, we need only show that $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^\alpha$ also holds.

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}$, then by Theorem 9.3 and the definition of $\mathcal{T}_{\mathcal{R}}$, we have

$$A \subseteq A^- \quad \text{and} \quad A \subseteq A^\circ.$$

Hence, by using the increasingness of \circ and $- \circ$, we can infer that

$$A^\circ \subseteq A^{\circ\circ} \quad \text{and} \quad A^- \subseteq A^{\circ\circ}.$$

Therefore, $A \subseteq A^{\circ\circ}$. Hence, by Definition 27.1, we can see that $A \in \mathcal{T}_{\mathcal{R}}^\alpha$. \square

Concerning reflexive relators, we can also easily prove the following

Theorem 27.6. *If \mathcal{R} is a reflexive relator on X , then*

$$(1) \quad \mathcal{T}_{\mathcal{R}}^\alpha \subseteq \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p; \quad (2) \quad \mathcal{T}_{\mathcal{R}}^s \cup \mathcal{T}_{\mathcal{R}}^p \subseteq \mathcal{T}_{\mathcal{R}}^\beta.$$

Proof. From Theorems 27.4 and 27.4, we can see that

$$\mathcal{T}_{\mathcal{R}}^s \cup \mathcal{T}_{\mathcal{R}}^p \subseteq \mathcal{T}_{\mathcal{R}}^\gamma \cup \mathcal{T}_{\mathcal{R}}^\delta \subseteq \mathcal{T}_{\mathcal{R}}^\beta,$$

and thus in particular (2) also holds. Therefore, we need only prove (1).

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^\alpha$, then by Definition 27.1 we have $A \subseteq A^{\circ\circ}$. Moreover, from Theorem 9.3, we can see $A^{\circ\circ} \subseteq A^{\circ-}$. Therefore, we also have $A \subseteq A^{\circ-}$. Hence, by Definition 19.2, we can see that $A \in \mathcal{T}_{\mathcal{R}}^s$.

Moreover, by Theorem 9.3, we also have $A^\circ \subseteq A$. Hence, by using the increasingness of the operation $- \circ$, we can infer that $A^{\circ\circ} \subseteq A^{\circ-}$. Therefore, we also have $A \subseteq A^{\circ-}$. Hence, by Definition 19.2, we can see that $A \in \mathcal{T}_{\mathcal{R}}^p$ also holds. Therefore, we actually have $A \in \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p$, and thus (1) is true. \square

Remark 27.7. Assertions (1) and (2) actually depend on the fact that if \mathcal{R} is a reflexive relator on X , then for any $A \subseteq X$ we have

$$A^{\circ\circ} \subseteq A^{\circ-} \cap A^{\circ-} \quad \text{and} \quad A^{\circ-} \cup A^{\circ-} \subseteq A^{\circ\circ}.$$

28. Characterizations of topologically α -open sets

Now, as a straightforward generalization of [112, Lemma 2.1] of Noiri and [128, Theorem 3] of Reilly and Vamanamurthy, we can also prove the following

Theorem 28.1. *If \mathcal{R} is a topological relator on X , then*

$$\mathcal{T}_{\mathcal{R}}^\alpha = \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p.$$

Proof. From Theorem 27.6, we know that $\mathcal{T}_{\mathcal{R}}^\alpha \subseteq \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p$ even if the relator \mathcal{R} is assumed to be only reflexive.

Moreover, if $A \in \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p$, i. e., $A \in \mathcal{T}_{\mathcal{R}}^s$ and $A \in \mathcal{T}_{\mathcal{R}}^p$, then by Definition 19.2, we have

$$A \subseteq A^{\circ-} \quad \text{and} \quad A \subseteq A^{\circ-}.$$

Hence, by using the increasingness of the operations $-$ and \circ , Theorems 10.3 and 10.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that

$$A^- \subseteq A^{\circ\circ-} \subseteq A^{\circ-} \quad \text{and thus} \quad A^{\circ-} \subseteq A^{\circ\circ-}.$$

Therefore, we also have $A \subseteq A^{\circ\circ-}$. Hence, by Definition 27.1, we can see that $A \in \mathcal{T}_{\mathcal{R}}^\alpha$ also holds even if \mathcal{R} is assumed to be only quasi-topological. \square

Moreover, as an improvement of [110, Proposition 4] of Njåstad, we can prove

Theorem 28.2. *If \mathcal{R} is a topologically filtered, topological relator on X , then for any $A \subseteq X$ the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$;
- (2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \setminus B$;
- (3) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \subseteq \text{res}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A))$ such that $A = V \setminus B$.

Proof. If (1) holds, then by Definition 27.1 we have $A \subseteq A^{\circ\circ}$, and thus

$$A = A^{\circ\circ} \setminus (A^{\circ\circ} \setminus A).$$

Hence, by defining

$$V = A^{\circ\circ} \quad \text{and} \quad B = A^{\circ\circ} \setminus A,$$

we can obtain $A = V \setminus B$. Moreover, by Theorems 10.3, 10.2 and 9.3, we can state that

$$V \in \mathcal{T}_{\mathcal{R}} \quad \text{and} \quad B \subseteq A^{\circ\circ} \setminus A^{\circ} = A^{\circ\ddagger}.$$

Therefore, (3) also holds.

To prove the implication (3) \implies (2), it is enough to note only that, by Theorems 10.3 and 10.2, we have $A^{\circ} \in \mathcal{T}_{\mathcal{R}}$. Thus, by Corollary 15.6, we have $A^{\circ\ddagger} \in \mathcal{N}_{\mathcal{R}}$. Therefore, if $B \subseteq A^{\circ\ddagger}$, then we also have $B \in \mathcal{N}_{\mathcal{R}}$.

Finally, if (2) holds, then by Theorems 12.5, 10.3, 10.2 and 9.3 we can see that

$$A^{\circ} = (V \setminus B)^{\circ} = (V \cap B^c)^{\circ} = V^{\circ} \cap B^{c^{\circ}} = V \cap B^{c^{\circ}}.$$

Hence, by using Corollary 14.4, we can infer that

$$V \cap B^{c^{\circ\circ}} \subseteq (V \cap B^{c^{\circ}})^{\circ} = A^{\circ\circ}.$$

Moreover, by using that $c \circ = -c$ and $c - = \circ c$, we can see that

$$B^{c^{\circ\circ}} = B^{-c^{\circ}} = B^{-\circ c} = \emptyset^c = X.$$

Therefore, we actually have $V \subseteq A^{\circ\circ}$, and hence also

$$A \subseteq V = V^{\circ} \subseteq A^{\circ\circ}.$$

Thus, by Definition 27.1, assertion (1) also holds. □

Remark 28.3. If \mathcal{R} is a topological relator on X , then by Theorems 10.3, 10.2 and 9.3, for any $A \subseteq X$, we have $A^{\circ\circ} = A^{\circ}$.

Therefore, by using the notation $A^{\ddagger} = \text{bnd}_{\mathcal{R}}(A)$, we can also state that $A^{\circ\ddagger} = A^{\circ\circ} \setminus A^{\circ} = A^{\circ\circ} \setminus A^{\circ\circ} = A^{\circ\ddagger}$.

Now, by using the above theorem, we can also prove the following improvement of [110, Corollary] of Njåstad.

Corollary 28.4. *For a nonvoid, topologically filtered, topological relator \mathcal{R} on X , the following assertions are equivalent :*

- (1) $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}$;
- (2) $\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}$;
- (3) $\mathcal{N}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$;
- (4) $\mathcal{N}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$.

Proof. From Theorem 27.5, we can see that $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha}$. Therefore, (1) and (2) are equivalent even if \mathcal{R} is only reflexive.

Moreover, if $A \in \mathcal{N}_{\mathcal{R}}$, then by using Theorem 9.3, the increasingness of \circ , and the definitions of $\mathcal{D}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$, we can see that $A^{\circ} \subseteq A^{-\circ} = \emptyset$, and thus $A^{\circ} = \emptyset$, i. e., $A \notin \mathcal{E}_{\mathcal{R}}$. Therefore, (3) and (4) are also equivalent even if \mathcal{R} is only reflexive.

From Theorem 10.3, we know that a topological relator is reflexive. Therefore, to complete the proof, we need only show that now (1) and (3) are also equivalent. For this, note that now $X \in \mathcal{T}_{\mathcal{R}}$ since $\mathcal{R} \neq \emptyset$. Therefore, if $B \in \mathcal{N}_{\mathcal{R}}$, then by Theorem 28.2 we also have $B^c = X \setminus B \in \mathcal{T}_{\mathcal{R}}^{\alpha}$. Hence, if (1) holds, we can infer that $B^c \in \mathcal{T}_{\mathcal{R}}$, and thus $B \in \mathcal{F}_{\mathcal{R}}$. Therefore, (1) implies (3).

On the other hand, if $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$, then by Theorem 28.2 there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \setminus B$. Moreover, if (3) holds, we can also state that $B \in \mathcal{F}_{\mathcal{R}}$, and thus $B^c \in \mathcal{T}_{\mathcal{R}}$. Hence, by using Corollary 12.6, we can infer that $A = V \setminus B = V \cap B^c \in \mathcal{T}_{\mathcal{R}}$. Therefore, (1) also holds. \square

29. Characterizations of topologically β -open sets

By using the plausible notation $\mathcal{F}_{\mathcal{R}}^r = \{A^c : A \in \mathcal{T}_{\mathcal{R}}^r\}$, as a partial counterpart of [54, Theorem 26] of Ekici and [77, Theorem 3.7] of Jamunarani et al., we can also prove the following

Theorem 29.1. *If \mathcal{R} is a topological relator on X , then for any $A \subseteq X$ the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$;
- (2) $\text{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^s$;
- (3) $\text{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^q$;
- (4) $\text{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}^r$.

Proof. From Theorem 20.1, we know that (2) and (3) are equivalent. Therefore, we need only prove the equivalence of (1), (2) and (4).

For this, note that if (1) holds, then by Definition 27.1 we have $A \subseteq A^{-\circ-}$. Hence, by using the increasingness of $-$, Theorems 10.3 and 10.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that

$$A^- \subseteq A^{-\circ--} \subseteq A^{-\circ-}.$$

Therefore, by Definition 19.2, assertion (2) also holds.

While, if (2) holds, then by using Theorems 10.3, 9.3, 10.2 and 20.2, we can see that

$$A^- = A^{---} = A^{-\circ-}.$$

Hence, by using that $-c = c \circ$ and $\circ c = c -$, we can infer that

$$A^{-c} = A^{-\circ-c} = A^{-\circ c \circ} = A^{-c \circ}.$$

Thus, by Definition 23.1, we have $A^c \in \mathcal{T}_{\mathcal{R}}^r$, and thus (4) also holds.

Finally, if (4) holds, then $A^c \in \mathcal{T}_{\mathcal{R}}^r$. Therefore, by Definition 23.1, we have $A^{-c} = A^{-c \circ}$. Hence, by using a similar argument as above, we can infer that $A^- = A^{-\circ-}$. Moreover, by Theorems 10.3 and 9.3, we also have $A \subseteq A^-$. Therefore, $A \subseteq A^{-\circ-}$, and thus by Definition 27.1 assertion (1) also holds. \square

Remark 29.2. From this theorem, by using our former results on the families $\mathcal{T}_{\mathcal{R}}^s$ and $\mathcal{T}_{\mathcal{R}}^r$, we can derive several properties of the family $\mathcal{T}_{\mathcal{R}}^{\beta}$.

For instance, from Theorem 29.1, by using Theorems 20.2 and 24.3, we can immediately derive the following

Theorem 29.3. *If \mathcal{R} is a topological relator on X , then for any $A \subseteq X$ the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$;
- (2) there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $\text{cl}_{\mathcal{R}}(A) = \text{cl}_{\mathcal{R}}(V)$;
- (3) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \subseteq X$ such that

$$\text{cl}_{\mathcal{R}}(A) = V \cup B \quad \text{and} \quad B \subseteq \text{res}_{\mathcal{R}}(V).$$

Hence, we can easily derive the following counterpart of [54, Theorem 27] of Ekici and [77, Theorem 3.8] of Jamunarani et al.

Corollary 29.4. If \mathcal{R} is a topological relator on X , $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ and

$$A \subseteq B \subseteq \text{cl}_{\mathcal{R}}(A),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{\beta}$ also holds.

Proof. By using the increasingness of $-$, Theorems 10.3 and 10.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can see that

$$A^{-} \subseteq B^{-} \subseteq A^{-\circ} \subseteq A^{-},$$

and thus $A^{-} = B^{-}$. Hence, by Theorem 29.3, it is clear that the required assertion is also true. \square

Moreover, from Theorem 29.1, by using Theorem 24.5, we can also easily derive

Theorem 29.5. If \mathcal{R} is a topologically filtered, topological relator on X and $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that

$$\text{cl}_{\mathcal{R}}(A) = V \cup B \quad \text{and} \quad V \cap B = \emptyset.$$

Now, analogously to [54, Theorem 23] of Ekici and [77, Theorem 3.5] of Jamunarani et al., we can also prove the following

Theorem 29.6. If \mathcal{R} is a topological relator on X and $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}^s$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$.

Proof. Define

$$V = A^{-} \quad \text{and} \quad B = A \cup A^{-c}.$$

Then, by Theorem 29.1, we have $V \in \mathcal{T}_{\mathcal{R}}^s$. Moreover, by using the increasingness of $-$ and Theorems 10.3 and 9.3, we can see that

$$B^{-} = (A \cup A^{-c})^{-} \subseteq A^{-} \cup A^{-c-} \subseteq A^{-} \cup A^{-c} = X.$$

Therefore, $B^{-} = X$, and thus $B \in \mathcal{D}_{\mathcal{R}}$.

Moreover, by using Theorems 10.3 and 9.3, we can also see that

$$V \cap B = A^{-} \cap (A \cup A^{-c}) = (A^{-} \cap A) \cup (A^{-} \cap A^{-c}) = A \cup \emptyset = A.$$

Therefore, the required assertion is also true. \square

The following example shows that the converse of this theorem is also false.

Example 29.7. If $X = \{1, 2, 3, 4\}$ and R_i is a relations on X , for every $i = 1, 2, 3$, such that

$$\begin{aligned} R_1(1) &= \{1\}, & R_1(2) &= R_1(3) = R_1(4) = X; \\ R_2(1) &= R_2(2) = \{1, 2\}, & R_2(3) &= R_2(4) = X; \\ R_3(1) &= X, & R_3(2) &= R_3(3) = \{2, 3\}, & R_3(4) &= X; \end{aligned}$$

then $\mathcal{R} = \{R_1, R_2, R_3\}$ is a preorder relator on X such that, under the notations

$$V = \{2, 3\} \quad \text{and} \quad B = \{1, 3, 4\},$$

we have $V \in \mathcal{T}_{\mathcal{R}}^s$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B \notin \mathcal{T}_{\mathcal{R}}^{\beta}$.

To check this, note that R_1 , R_2 and R_3 are just the Pervin preorders generated by the sets $\{1\}$, $\{1, 2\}$ and $\{2, 3\}$, respectively. Moreover, we have

$$A^{-\circ} = \{3\}^{-\circ} = \{3, 4\}^{-\circ} = \emptyset^{-} = \emptyset,$$

$$V^{\circ} = \{2, 3\}^{\circ} = \{2, 3\}^{-} = \{2, 3, 4\} \quad \text{and} \quad B^{-} = \{1, 3, 4\}^{-} = X.$$

30. Some further results on topologically α -open and β -open sets

By using our former results and the plausible notation $\mathcal{F}_{\mathcal{R}}^{\kappa} = \{A^c : A \in \mathcal{T}_{\mathcal{R}}^{\kappa}\}$ with $\kappa = s$ and $\kappa = \beta$, we can also easily prove the following counterpart of [54, Theorem 7] of Ekici and [77, Theorem 2.1] of Jamunarani et al.

Theorem 30.1. *If \mathcal{R} is a topological relator on X , then*

$$\mathcal{T}_{\mathcal{R}}^r = \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta}.$$

Proof. By Theorem 23.6, we have $\mathcal{T}_{\mathcal{R}}^r = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^s$. Moreover, by Theorems 27.5 and 27.6, we have

$$\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha} \quad \text{and} \quad \mathcal{T}_{\mathcal{R}}^s \subseteq \mathcal{T}_{\mathcal{R}}^{\beta},$$

and thus also $\mathcal{F}_{\mathcal{R}}^s \subseteq \mathcal{F}_{\mathcal{R}}^{\beta}$. Therefore, $\mathcal{T}_{\mathcal{R}}^r \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta}$.

On the other hand, if $A \in \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta}$, then we have

$$A \in \mathcal{T}_{\mathcal{R}}^{\alpha} \quad \text{and} \quad A \in \mathcal{F}_{\mathcal{R}}^{\beta},$$

and thus also $A^c \in \mathcal{T}_{\mathcal{R}}^{\beta}$. Hence, by using Definition 27.1, we can infer that

$$A \subseteq A^{\circ\circ} \quad \text{and} \quad A^c \subseteq A^{c\circ\circ},$$

and thus also $A^{c\circ\circ} \subseteq A$. Moreover, by using the equalities $c- = \circ c$, $-c = c\circ$ and $-- = c\circ c$, we can see that

$$A^{c\circ\circ} = A^{\circ c\circ} = A^{\circ c\circ c} = A^{\circ\circ}.$$

Therefore, $A^{\circ\circ} \subseteq A$, and thus $A = A^{\circ\circ}$ also holds. Hence, by using Theorem 23.10, we can already infer that $A \in \mathcal{T}_{\mathcal{R}}^r$. Therefore, $\mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta} \subseteq \mathcal{T}_{\mathcal{R}}^r$, and thus the required equality is also true. \square

Now, by using Theorem 27.3 and 29.1, we can also easily prove the following two counterparts of [8, Theorem 2.4] of Andrijević.

Theorem 30.2. *If \mathcal{R} is a topological relator on X , then $\mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{T}_{\mathcal{R}}^{\beta}$.*

Proof. By Theorem 27.3, we always have $\mathcal{T}_{\mathcal{R}}^{\gamma} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$. Therefore, we need only prove that now $\mathcal{T}_{\mathcal{R}}^{\beta} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma}$ also holds.

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then by Theorem 29.1 we have $A^- \in \mathcal{T}_{\mathcal{R}}^s$. Hence, by defining $V = A^-$, we can note that $V \in \mathcal{T}_{\mathcal{R}}^s$ such that $V \subseteq A^-$. Moreover, by Theorems 10.3 and 9.3, it is clear that $A \subseteq V$ is also true. Therefore, by Definition 27.1 we also have $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$. \square

Theorem 30.3. *If \mathcal{R} is a topologically filtered, topological relator on X , then $\mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{T}_{\mathcal{R}}^{\beta}$.*

Proof. By Theorem 27.3, we always have $\mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$. Therefore, we need only prove that now $\mathcal{T}_{\mathcal{R}}^{\beta} \subseteq \mathcal{T}_{\mathcal{R}}^{\delta}$ also holds.

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then by Theorem 29.1 we have $A^- \in \mathcal{T}_{\mathcal{R}}^s$. Hence, by using Theorems 10.3, 9.3, 10.2 and 20.2, we can again infer that

$$A^- = A^{--} = A^{-\circ}.$$

Now, by defining $V = A \cap A^{-\circ}$, we can note that $V \subseteq A$. Moreover, by using Corollary 14.4 and Theorems 10.3, 10.2 and 9.3, we can see that

$$V^- = (A \cap A^{-\circ})^- \supseteq A^- \cap A^{-\circ} = A^{-\circ}.$$

Hence, we can infer that

$$V^{-\circ} \supseteq A^{-\circ\circ} = A^{-\circ} \supseteq A \cap A^{-\circ} = V.$$

Thus, by Definition 19.2, we also have $V \in \mathcal{T}_{\mathcal{R}}^{\delta}$.

Moreover, we can now also note that

$$A \subseteq A^- = A^{-\circ} \subseteq V^{-\circ} = V^-.$$

Therefore, by Definition 27.1, we also have $A \in \mathcal{T}_{\mathcal{R}}^{\delta}$. \square

Unfortunately, concerning the relationship of the families $\mathcal{T}_{\mathcal{R}}^{\alpha}$ and $\mathcal{T}_{\mathcal{R}}^{\gamma}$, we can only prove the following

Theorem 30.4. *If \mathcal{R} is a reflexive relator on X , then $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma} \cap \mathcal{T}_{\mathcal{R}}^{\delta}$.*

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$, then by Theorem 27.6 we also have $A \in \mathcal{T}_{\mathcal{R}}^s$ and $A \in \mathcal{T}_{\mathcal{R}}^p$. Hence, by Theorem 27.4, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ and $A \in \mathcal{T}_{\mathcal{R}}^{\delta}$ also hold. Therefore, the required inclusion is also true. \square

Remark 30.5. Later, we shall see that the corresponding equality need not be true. Moreover, $\mathcal{T}_{\mathcal{R}}^{\delta}$ may also be a proper subset of $\mathcal{T}_{\mathcal{R}}^{\gamma}$.

31. Topologically a -open and b -open sets

In topological spaces, β -open sets were actually called semi-preopen by Andrijević [8]. Later, this terminology was also used by Ganster and Andrijević [60] and Dontchev [48].

In a subsequent paper [12], Andrijević also introduced the notion of a b -open subset of a topological space. Motivated by his definition, we may also naturally introduce the following

Definition 31.1. A subset A of a relator space $X(\mathcal{R})$ will be called *topologically*

- (1) *a -open* if $A \subseteq \text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A)) \cap \text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(A))$;
- (2) *b -open* if $A \subseteq \text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A)) \cup \text{int}_{\mathcal{R}}(\text{cl}_{\mathcal{R}}(A))$.

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^a$ and $\mathcal{T}_{\mathcal{R}}^b$, respectively.

Thus, analogously to Theorem 27.6, we evidently have the following

Theorem 31.2. *We have*

- (1) $\mathcal{T}_{\mathcal{R}}^a = \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p$;
- (2) $\mathcal{T}_{\mathcal{R}}^s \cup \mathcal{T}_{\mathcal{R}}^p \subseteq \mathcal{T}_{\mathcal{R}}^b$.

Moreover, by using Theorems 27.6 and 9.3 and Definition 27.1, we can also prove

Theorem 31.3. *If \mathcal{R} is reflexive relator on X , then*

- (1) $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^a$
- (2) $\mathcal{T}_{\mathcal{R}}^b \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$.

Proof. Since (1) follows immediately from Theorems 27.6 and 31.2, we need only prove (2).

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^b$, then by Definition 31.1 and Remark 27.7, we have $A \subseteq A^{\circ-} \cup A^{-\circ} \subseteq A^{-\circ-}$. Thus, by Definition 27.1, $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ also holds. \square

Remark 31.4. Note that, by Theorem 31.2, we always have $\mathcal{T}_{\mathcal{R}}^a \subseteq \mathcal{T}_{\mathcal{R}}^b$. Thus, if \mathcal{R} is reflexive relator on X , then by Theorem 31.3 we also have $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$.

Now, in accordance with [12, Remark 1] of Andrijević, we can also prove

Theorem 31.5. *If \mathcal{R} is a topologically filtered, topological relator on X , then for any $A \subseteq X$ the following assertions are equivalent:*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^b$;
- (2) there exist $B \in \mathcal{T}_{\mathcal{R}}^s$ and $C \in \mathcal{T}_{\mathcal{R}}^p$ such that $A = B \cup C$.

Proof. If (2) holds then by Definition 19.2 we have

$$B \subseteq B^{\circ-} \quad \text{and} \quad C \subseteq C^{-\circ}.$$

Moreover, we can see that $B \subseteq A$ and $C \subseteq A$. Hence, by using the increasingness of $\circ-$ and $-\circ$, we can infer that

$$B^{\circ-} \subseteq A^{\circ-} \quad \text{and} \quad C^{-\circ} \subseteq A^{-\circ}.$$

Therefore, we have

$$A = B \cup C \subseteq A^{\circ-} \cup A^{-\circ}.$$

Thus, by Definition 31.1, assertion (1) also holds even if \mathcal{R} is not assumed to have any particular properties.

Conversely, if (1) holds, then by Definition 31.1, we have $A \subseteq A^{\circ-} \cup A^{-\circ}$. Hence, we can infer that

$$A = A \cap (A^{\circ-} \cup A^{-\circ}) = (A \cap A^{\circ-}) \cup (A \cap A^{-\circ}).$$

Thus, by defining

$$B = A \cap A^{\circ-} \quad \text{and} \quad C = A \cap A^{-\circ},$$

we can at once state that $A = B \cup C$.

Now, by using Theorems 12.5, 10.3, 10.2 and 9.3, we can also see that

$$B^{\circ} = (A \cap A^{\circ-})^{\circ} = A^{\circ} \cap A^{\circ\circ-} \supseteq A^{\circ} \cap A^{\circ\circ} = A^{\circ} \cap A^{\circ} = A^{\circ},$$

and thus

$$B^{\circ-} \supseteq A^{\circ-} \supseteq A \cap A^{\circ-} = B.$$

Therefore, by Definition 19.2, we have $B \in \mathcal{T}_{\mathcal{R}}^s$.

Moreover, by using Corollary 14.4, Theorem 12.5, 10.3, 10.2 and 9.3, we can also see that

$$C^{-} = (A \cap A^{-\circ})^{-} \supseteq A^{-} \cap A^{-\circ} = A^{-\circ},$$

and thus

$$C^{-\circ} \supseteq A^{-\circ\circ} = A^{-\circ} \supseteq A \cap A^{-\circ} = C.$$

Therefore, by Definition 19.2, we also have $C \in \mathcal{T}_{\mathcal{R}}^p$. □

Remark 31.6. The above proof shows that if \mathcal{R} is a topologically filtered, topological relator on X , then for any $A \subseteq X$ we have

$$(1) \ A \cap A^{\circ-} \in \mathcal{T}_{\mathcal{R}}^s; \quad (2) \ A \cap A^{-\circ} \in \mathcal{T}_{\mathcal{R}}^p.$$

32. Generalized topologically open sets derived from topologically simple relators

By Definition 19.2 and Theorem 18.6, we can at once state the following

Theorem 32.1. *For a nonvoid relator \mathcal{R} on X , the following assertions are equivalent :*

$$(1) \ \mathcal{T}_{\mathcal{R}}^p = \mathcal{P}(X); \quad (2) \ \mathcal{R} \text{ is a properly topologically symmetric.}$$

Hence, by using Theorems 23.4 and 31.2, we can immediately derive

Corollary 32.2. *If \mathcal{R} is a nonvoid, properly topologically symmetric relator on X , then*

$$(1) \ \mathcal{T}_{\mathcal{R}}^r = \mathcal{F}_{\mathcal{R}}^s; \quad (2) \ \mathcal{T}_{\mathcal{R}}^a = \mathcal{T}_{\mathcal{R}}^s; \quad (3) \ \mathcal{T}_{\mathcal{R}}^b = \mathcal{P}(X).$$

Proof. Namely, by Theorems 23.4 and 31.2 and 32.1, we have

$$\mathcal{T}_{\mathcal{R}}^r = \mathcal{T}_{\mathcal{R}}^p \cap \mathcal{F}_{\mathcal{R}}^s = \mathcal{P}(X) \cap \mathcal{F}_{\mathcal{R}}^s = \mathcal{F}_{\mathcal{R}}^s$$

and

$$\mathcal{T}_{\mathcal{R}}^a = \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{T}_{\mathcal{R}}^p = \mathcal{T}_{\mathcal{R}}^s \cap \mathcal{P}(X) = \mathcal{T}_{\mathcal{R}}^s.$$

Moreover, by Theorems 31.2 and 32.1, we also have

$$\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}}^s \cup \mathcal{P}(X) = \mathcal{T}_{\mathcal{R}}^s \cup \mathcal{T}_{\mathcal{R}}^p \subseteq \mathcal{T}_{\mathcal{R}}^b,$$

and thus $\mathcal{T}_{\mathcal{R}}^b = \mathcal{P}(X)$ also holds. □

Remark 32.3. By using equality (1), we can also easily see that

$$A \in \mathcal{F}_{\mathcal{R}}^r \iff A^c \in \mathcal{T}_{\mathcal{R}}^r \iff A^c \in \mathcal{F}_{\mathcal{R}}^s \iff A \in \mathcal{T}_{\mathcal{R}}^s.$$

Therefore, in addition to Corollary 32.2, we can also state that $\mathcal{F}_{\mathcal{R}}^r = \mathcal{T}_{\mathcal{R}}^s$.

Now, in addition to Corollary 20.4, we can also state the following

Corollary 32.4. *If \mathcal{R} is a nonvoid, properly topologically symmetric relator on X , then for any $A \subseteq X$, the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^s$; (2) $A = \text{cl}_{\mathcal{R}}(\text{int}_{\mathcal{R}}(A))$.

Proof. By Remark 32.3 and the equalities $c - = \circ c$ and $c \circ c = -$, we have

$$A \in \mathcal{T}_{\mathcal{R}}^s \iff A \in \mathcal{F}_{\mathcal{R}}^r \iff A^c \in \mathcal{T}_{\mathcal{R}}^r \iff A^c = A^{c \circ -} \iff A = A^{c \circ - \circ c} \iff A = A^{\circ c \circ c} \iff A = A^{\circ -}.$$

Moreover, from Theorem 32.1, by Theorems 20.1 and 28.1, we can also derive □

Corollary 32.5. *If \mathcal{R} is a nonvoid, properly topologically symmetric, topological relator on X , then*

- (1) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{P}(X)$; (2) $\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}^s$.

From Corollary 17.10, we know that a relator \mathcal{R} is properly topologically symmetric if and only if it is topologically simple and weakly symmetric.

Therefore, as a straightforward generalization of properly symmetric relators, we may also naturally consider here topologically simple relators.

Theorem 32.6. *If \mathcal{R} is a topologically simple relator on X and $R = \bigcap \mathcal{R}$, then for any $A \subseteq X$ we have*

- (1) $A \in \mathcal{T}_{\mathcal{R}}^s$ if and only if $A \subseteq R^{-1}[R^{-1}[A^c]^c]$;
 (2) $A \in \mathcal{T}_{\mathcal{R}}^p$ if and only if $A \subseteq R^{-1}[R^{-1}[A]^c]^c$;
 (3) $A \in \mathcal{T}_{\mathcal{R}}^q$ if and only if there exists $V \subseteq X$ such that

$$R[V] \subseteq V \subseteq A \subseteq R^{-1}[V].$$

 (4) $A \in \mathcal{T}_{\mathcal{R}}^{ps}$ if and only if there exists $V \subseteq X$ such that

$$A \cup R[V] \subseteq V \subseteq R^{-1}[A].$$

Proof. By Theorem 16.5, we have $\mathcal{R}^{\wedge} = \{R\}^{\wedge}$. Hence, by using Theorems 6.4 and 4.9, we can see that

$$\text{cl}_{\mathcal{R}}(A) = \text{cl}_R(A) = R^{-1}[A] \quad \text{and} \quad \text{int}_{\mathcal{R}}(A) = \text{int}_R(A) = R^{-1}[A^c]^c$$

for all $A \subseteq X$. Moreover, by the corresponding definitions and Theorem 6.4, we can also see that

$$A \in \mathcal{T}_{\mathcal{R}} \iff A \subseteq \text{int}_{\mathcal{R}}(A) \iff A \subseteq \text{int}_R(A) \iff R[A] \subseteq A.$$

Hence, by Definition 19.2, it is clear that required assertions are true. □

Remark 32.7. By Definitions 23.1 and 27.1, in addition to assertions (1)–(4), we can, for instance, also state that

- (a) $A \in \mathcal{T}_{\mathcal{R}}^r$ if and only if $A = R^{-1}[R^{-1}[A]^c]^c$;
 (b) $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ if and only if there exists $V \subseteq X$ such that

$$A \subseteq V \subseteq R^{-1}[A] \cap R^{-1}[R^{-1}[A^c]^c].$$

By using our former results on the various refinements of relators, we can also easily prove the following

Theorem 32.8. *For any $\square = *, \#$ or \wedge and $\kappa = s, p, q, ps, r, \alpha, \beta, \gamma, \delta, a$ or b , we have*

$$\mathcal{T}_{\mathcal{R}}^{\kappa} = \mathcal{T}_{\mathcal{R}^{\square}}^{\kappa}.$$

Proof. By Theorems 6.7 and 6.4, we have $\mathcal{R}^\wedge = \mathcal{R}^{\square^\wedge}$. Thus, by Theorem 6.4 and the definition of $\mathcal{T}_{\mathcal{R}}$, we have

$$\text{cl}_{\mathcal{R}} = \text{cl}_{\mathcal{R}^{\square}}, \quad \text{int}_{\mathcal{R}} = \text{int}_{\mathcal{R}^{\square}} \quad \text{and} \quad \mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}.$$

Hence, by Definitions 19.2, 23.1, 27.1 and 31.1, it is clear that the required equality is also true. \square

Remark 32.9. By [110, 29, 31, 27, 13, 10, 11], for any two relators \mathcal{R} and \mathcal{S} on X , the possible consequences and equivalents of the inclusion $\mathcal{T}_{\mathcal{R}}^{\kappa} \subseteq \mathcal{T}_{\mathcal{S}}^{\kappa}$ should also be investigated.

33. Topologically semi-open and preopen sets derived from the paratopological refinements of relators

In addition to Theorem 32.8, we can also prove the following

Theorem 33.1. *If \mathcal{R} is a non-degenerated relator on X , then for any $A \subseteq X$ the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}^\Delta}^s \setminus \{\emptyset\}$; (2) $A \in \mathcal{E}_{\mathcal{R}}$ and \mathcal{R} is non-partial.

Proof. By Definition 19.2, assertion (1) is equivalent to the statement that

- (a) $A \subseteq \text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A))$ and $A \neq \emptyset$.

Therefore, if (1) holds, then in particular we have

- (b) $\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A)) \neq \emptyset$.

Next, we show that (b) already implies (2). For this, note that if $A \notin \mathcal{E}_{\mathcal{R}}$, then by Corollary 7.8 we have $\text{int}_{\mathcal{R}^\Delta}(A) = \emptyset$. Therefore, if (b) holds, then

$$\text{cl}_{\mathcal{R}^\Delta}(\emptyset) = \text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A)) \neq \emptyset.$$

Hence, by using Corollary 7.8, we can infer that $\emptyset \in \mathcal{D}_{\mathcal{R}}$. Thus, by Theorem 9.10, the relator \mathcal{R} cannot be non-degenerated. This contradiction proves that (b) implies $A \in \mathcal{E}_{\mathcal{R}}$.

However, if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 7.8 we have $\text{int}_{\mathcal{R}^\Delta}(A) = X$. Therefore, if (b) holds, then

$$\text{cl}_{\mathcal{R}^\Delta}(X) = \text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A)) \neq \emptyset.$$

Hence, by using Corollary 7.8, we can infer that $X \in \mathcal{D}_{\mathcal{R}}$. Thus, by Theorem 9.8, the relator \mathcal{R} is non-partial. Consequently, (b) implies (2), and thus (1) also implies (2).

In the sequel, we shall show that (2) implies that

- (c) $\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A)) = X$ and $A \neq \emptyset$.

For this, note that if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 7.8, we have $\text{int}_{\mathcal{R}^\Delta}(A) = X$. Moreover, if \mathcal{R} is non-partial, then by Theorem 9.8 we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ and $X \in \mathcal{D}_{\mathcal{R}}$. Thus, in particular $A \neq \emptyset$. Moreover, by Corollary 7.8, we have $\text{cl}_{\mathcal{R}^\Delta}(X) = X$. Therefore, if (2) holds, then in addition to $A \neq \emptyset$ we also have

$$\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A)) = \text{cl}_{\mathcal{R}^\Delta}(X) = X,$$

and thus (c) also holds.

Hence, since (2) \implies (c) \implies (a) \implies (1), we can see that (2) also implies (1). \square

Now, by using the above theorem, we can also easily establish

Corollary 33.2. *If \mathcal{R} is a non-partial, non-degenerated relator on X , then*

$$\mathcal{T}_{\mathcal{R}^\Delta}^s = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \quad \text{with} \quad \emptyset \notin \mathcal{E}_{\mathcal{R}}.$$

Proof. To prove this, note that by Theorem 33.1 we have $\mathcal{T}_{\mathcal{R}^\Delta}^s \setminus \{\emptyset\} = \mathcal{E}_{\mathcal{R}}$. Moreover, by Definition 19.2, we have $\emptyset \in \mathcal{T}_{\mathcal{R}^\Delta}^s$ for any relator \mathcal{R} on X . And, by and Theorem 9.8, we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ for any non-partial relator \mathcal{R} on X . \square

Concerning the family $\mathcal{T}_{\mathcal{R}^\Delta}^p$, instead of an analogue of Theorem 33.1, we can only prove the following counterpart of Corollary 33.2.

Theorem 33.3. *If \mathcal{R} is a non-partial, non-degenerated relator on X , then*

$$\mathcal{T}_{\mathcal{R}^\Delta}^p = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \text{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}^\Delta}^p \setminus \{\emptyset\}$, then by Definition 19.2 we have

(a) $A \subseteq \text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(A))$ and $A \neq \emptyset$.

Thus, in particular we have

(b) $\text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(A)) \neq \emptyset$.

Next, we show that (b) already implies $A \in \mathcal{D}_{\mathcal{R}}$. For this, note that if $A \notin \mathcal{D}_{\mathcal{R}}$, then by Corollary 7.8 we have $\text{cl}_{\mathcal{R}^\Delta}(A) = \emptyset$. Therefore, if (b) holds, then

$$\text{int}_{\mathcal{R}^\Delta}(\emptyset) = \text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(A)) \neq \emptyset.$$

Hence, by using Corollary 7.8, we can infer that $\emptyset \in \mathcal{E}_{\mathcal{R}}$. Thus, by Theorem 9.8, the relator \mathcal{R} cannot be non-partial. This contradiction proves that (b) implies $A \in \mathcal{D}_{\mathcal{R}}$, and thus $A \in \mathcal{T}_{\mathcal{R}^\Delta}^p \setminus \{\emptyset\}$ also implies $A \in \mathcal{D}_{\mathcal{R}}$.

In the sequel, we shall show that if $A \in \mathcal{D}_{\mathcal{R}}$, then

(c) $\text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(A)) = X$ and $A \neq \emptyset$.

For this, note that if $A \in \mathcal{D}_{\mathcal{R}}$, then by Corollary 7.8 we have $\text{cl}_{\mathcal{R}^\Delta}(A) = X$. Moreover, by Theorems 9.10 and 9.8, we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$ and $X \in \mathcal{D}_{\mathcal{R}}$. Thus, in particular $A \neq \emptyset$. Moreover, by Corollary 7.8, we have $\text{cl}_{\mathcal{R}^\Delta}(X) = X$. Therefore, in addition to $A \neq \emptyset$, we also have

$$\text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(A)) = \text{int}_{\mathcal{R}^\Delta}(X) = X,$$

and thus (c) also holds. Hence, since $A \in \mathcal{D}_{\mathcal{R}} \implies (c) \implies (a) \implies A \in \mathcal{T}_{\mathcal{R}^\Delta}^p \setminus \{\emptyset\}$, we can see that $A \in \mathcal{D}_{\mathcal{R}}$ also implies $A \in \mathcal{T}_{\mathcal{R}^\Delta}^p \setminus \{\emptyset\}$.

Thus, we have proved that $\mathcal{T}_{\mathcal{R}^\Delta}^p \setminus \{\emptyset\} = \mathcal{D}_{\mathcal{R}}$. Therefore, to obtain the required assertion, we need only note that, by Definition 19.2, we have $\emptyset \in \mathcal{T}_{\mathcal{R}^\Delta}^p$ for any relator \mathcal{R} on X . Moreover, by Theorem 9.10 we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$ for any non-degenerated relator \mathcal{R} on X . \square

From Theorem 33.3 and Corollary 33.2, by using Theorem 23.4, we can derive

Theorem 33.4. *If \mathcal{R} is a non-partial, non-degenerated relator on X , then $\mathcal{T}_{\mathcal{R}^\Delta}^r = \{\emptyset, X\}$.*

Proof. If $A \in \mathcal{T}_{\mathcal{R}^\Delta}^r$, then by Theorem 23.4 we have

$$A \in \mathcal{T}_{\mathcal{R}^\Delta}^p \quad \text{and} \quad A \in \mathcal{F}_{\mathcal{R}^\Delta}^s,$$

and thus $A^c \in \mathcal{T}_{\mathcal{R}^\Delta}^s$. Hence, by using Theorem 33.3 and Corollary 33.2, we can infer that

$$(A \in \mathcal{D}_{\mathcal{R}} \quad \text{or} \quad A = \emptyset) \quad \text{and} \quad (A^c \in \mathcal{E}_{\mathcal{R}} \quad \text{or} \quad A^c = \emptyset),$$

and thus $A \notin \mathcal{D}_{\mathcal{R}}$ or $A = X$. Therefore, if $A \in \mathcal{D}_{\mathcal{R}}$, then only $A = X$ can hold. Hence, we can see that if $A \in \mathcal{T}_{\mathcal{R}^\Delta}^r$, then either $A = X$ or $A = \emptyset$ holds. Thus, $\mathcal{T}_{\mathcal{R}^\Delta}^r \subseteq \{\emptyset, X\}$.

On the other hand, from Theorems 33.3 and 9.8, we can see that $\emptyset, X \in \mathcal{T}_{\mathcal{R}^\Delta}^p$. Moreover, from Corollary 33.2 and Theorem 9.10, we can see that $\emptyset, X \in \mathcal{T}_{\mathcal{R}^\Delta}^s$, and thus also $\emptyset, X \in \mathcal{F}_{\mathcal{R}^\Delta}^s$. Therefore, by Theorem 23.4, we also have $\emptyset, X \in \mathcal{T}_{\mathcal{R}^\Delta}^r$, and thus also $\{\emptyset, X\} \subseteq \mathcal{T}_{\mathcal{R}^\Delta}^r$. \square

34. Topologically quasi-open and pseudo-open sets derived from the paratopological refinements of relators

In addition to Theorem 33.1, we can also prove the following

Theorem 34.1. *If \mathcal{R} is a non-degenerated relator on X , then for any $A \subseteq X$ the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}^\Delta}^q \setminus \{\emptyset\}$;
- (2) *there exists $V \in \mathcal{E}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}}$ such that $V \subseteq A$.*

Proof. If (1) holds, then $A \neq \emptyset$. Moreover, by Definition 19.2, there exists $V \subseteq X$ such that

$$V \in \mathcal{T}_{\mathcal{R}^\Delta} \quad \text{and} \quad V \subseteq A \subseteq \text{cl}_{\mathcal{R}^\Delta}(V).$$

Thus, in particular $\text{cl}_{\mathcal{R}^\Delta}(V) \neq \emptyset$. Hence, by using Corollary 7.8, we can infer that $V \in \mathcal{D}_{\mathcal{R}}$. Moreover, from Theorem 9.10, we can see that $\emptyset \notin \mathcal{D}_{\mathcal{R}}$, and thus $V \neq \emptyset$. Furthermore, from Corollary 7.9, we can see that $\mathcal{T}_{\mathcal{R}^\Delta} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$. Therefore, we necessarily have $V \in \mathcal{E}_{\mathcal{R}}$, and thus (2) also holds.

On the other hand, if (2) holds, then we have both $V \in \mathcal{E}_{\mathcal{R}}$ and $V \in \mathcal{D}_{\mathcal{R}}$. Hence, by Corollary 7.9, we can see that $V \in \mathcal{T}_{\mathcal{R}^\Delta}$. Moreover, from Theorem 9.10, we can see that $V \neq \emptyset$, and thus $A \neq \emptyset$. Furthermore, from Corollary 7.8, we can see that $\text{cl}_{\mathcal{R}^\Delta}(V) = X$, and thus $A \subseteq \text{cl}_{\mathcal{R}^\Delta}(V)$ trivially holds. Therefore, by Definition 19.2, assertion (1) also holds. \square

Now, by using the above theorem, we can also easily establish

Corollary 34.2. *If \mathcal{R} is a non-degenerated relator on X , then*

$$\mathcal{T}_{\mathcal{R}^\Delta}^q = \mathcal{E}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \text{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Proof. To derive this from Theorem 34.1, note that $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are ascending families of subsets of X . Therefore, if assertion (2) in Theorem 34.1 holds, then we necessarily have $A \in \mathcal{E}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}}$. Moreover, by Definition 19.2, we have $\emptyset \in \mathcal{T}_{\mathcal{R}^\Delta}^q$. And, by Theorem 9.10, we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$. \square

Analogously, to Theorem 34.1, we can also prove

Theorem 34.3. *If \mathcal{R} is a non-degenerated relator on X , then for any $A \subseteq X$ the following assertions are equivalent :*

- (1) $A \in \mathcal{T}_{\mathcal{R}^\Delta}^{ps} \setminus \{\emptyset\}$;
- (2) $A \in \mathcal{D}_{\mathcal{R}}$ and there exists $V \in \mathcal{E}_{\mathcal{R}}$ such that $A \subseteq V$.

Proof. If (1) holds, then $A \neq \emptyset$. Moreover, by Definition 19.2, there exists $V \subseteq X$ such that

$$V \in \mathcal{T}_{\mathcal{R}^\Delta} \quad \text{and} \quad A \subseteq V \subseteq \text{cl}_{\mathcal{R}^\Delta}(A).$$

Thus, in particular $V \neq \emptyset$ and $\text{cl}_{\mathcal{R}^\Delta}(A) \neq \emptyset$. Hence, by using Corollary 7.8, we can infer that $A \in \mathcal{D}_{\mathcal{R}}$. Moreover, from Corollary 7.9, we can see that $\mathcal{T}_{\mathcal{R}^\Delta} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$. Therefore, $V \in \mathcal{E}_{\mathcal{R}}$, and thus (2) also holds.

On the other hand, if (2) holds, then by Theorem 9.10 and Corollary 7.8 we have $A \neq \emptyset$ and $\text{cl}_{\mathcal{R}^\Delta}(A) = X$. Thus, we trivially have $V \subseteq \text{cl}_{\mathcal{R}^\Delta}(A)$. Moreover, from Corollary 7.9, we can see that $V \in \mathcal{T}_{\mathcal{R}^\Delta}$. Therefore, by Definition 19.2, assertion (1) also holds. \square

Now, by using the above theorem, we can also easily establish

Corollary 34.4. *If \mathcal{R} is a non-degenerated relator on X , then*

$$\mathcal{T}_{\mathcal{R}^\Delta}^{ps} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \text{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Proof. To derive this from Theorem 34.3, note that, by Definition 19.2, for any relator \mathcal{R} on X we have $\emptyset \in \mathcal{T}_{\mathcal{R}^\Delta}^{ps}$. Moreover, by Theorem 9.10, we have $X \in \mathcal{E}_{\mathcal{R}}$. Therefore, in assertion (2) of Theorem 34.3, we can take $V = X$. Furthermore, by Theorem 9.10, we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$. \square

Remark 34.5. If \mathcal{R} is a non-partial, non-degenerated relator on X , then by Theorem 33.3 and Corollary 34.4 we have $\mathcal{T}_{\mathcal{R}^\Delta}^p = \mathcal{T}_{\mathcal{R}^\Delta}^{ps}$.

However, the following theorem shows that, for a non-degenerated relator \mathcal{R} on X , the above equality need not be true.

Theorem 34.6. *If \mathcal{R} is a partial relator on X , then $\mathcal{T}_{\mathcal{R}^\Delta}^{ps} = \{\emptyset\}$ and $\mathcal{T}_{\mathcal{R}^\Delta}^p = \mathcal{P}(X)$.*

Proof. By Theorem 9.8, we have $\emptyset \in \mathcal{E}_{\mathcal{R}}$ and $X \notin \mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{D}_{\mathcal{R}}$ is an ascending family of subsets of X , we can see that, for any $A \subseteq X$, we also have $A \notin \mathcal{D}_{\mathcal{R}}$. Now, by using Corollary 7.8, we can see that

$$\text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(A)) = \text{int}_{\mathcal{R}^\Delta}(\emptyset) = X.$$

Thus, by Definition 19.2, we have $\mathcal{T}_{\mathcal{R}^\Delta}^p = \mathcal{P}(X)$. Moreover, by Corollary 34.4, we can also see that

$$\mathcal{T}_{\mathcal{R}^\Delta}^{ps} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} = \emptyset \cup \{\emptyset\} = \{\emptyset\}.$$

Namely, since \mathcal{R} is a partial relator on X , there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) = \emptyset$. Thus, in particular $X \neq \emptyset$ and $\mathcal{R} \neq \emptyset$. Therefore, \mathcal{R} is a non-degenerated relator on X . \square

35. Topologically α -open, β -open, γ -open and δ -open sets derived from the paratopological refinements of relators

By using Theorems 9.8 and 9.10 and Corollary 7.8, we can also prove

Theorem 35.1. *If \mathcal{R} is a non-partial, non-degenerated relator on X , then*

$$\mathcal{T}_{\mathcal{R}^\Delta}^\alpha = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \quad \text{with} \quad \emptyset \notin \mathcal{E}_{\mathcal{R}}.$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}^\Delta}^\alpha$, then by Definition 27.1 we have

$$A \subseteq \text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A))).$$

Hence, if $A \neq \emptyset$, we can infer that

$$\text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A))) \neq \emptyset.$$

Therefore, by Corollary 7.8, we necessarily have

$$\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A)) \in \mathcal{E}_{\mathcal{R}}.$$

Moreover, from Theorem 9.8, we can see that $\emptyset \notin \mathcal{E}_{\mathcal{R}}$, and thus

$$\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A)) \neq \emptyset.$$

Hence, by using Corollary 7.8, we can infer that $\text{int}_{\mathcal{R}^\Delta}(A) \in \mathcal{D}_{\mathcal{R}}$. Moreover, from Theorem 9.10, we can see that $\emptyset \notin \mathcal{D}_{\mathcal{R}}$, and thus $\text{int}_{\mathcal{R}^\Delta}(A) \neq \emptyset$. Hence, by using Corollary 7.8, we can already infer that $A \in \mathcal{E}_{\mathcal{R}}$. Thus, we have proved that $\mathcal{T}_{\mathcal{R}^\Delta}^\alpha \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, and hence $\mathcal{T}_{\mathcal{R}^\Delta}^\alpha \subseteq \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$.

On the other hand, if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 7.8 we have $\text{int}_{\mathcal{R}^\Delta}(A) = X$. Moreover, from Theorem 9.8, we can see that $X \in \mathcal{D}_{\mathcal{R}}$. Hence, by using Corollary 7.8, we can infer that

$$\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A)) = \text{cl}_{\mathcal{R}^\Delta}(X) = X.$$

Moreover, from Theorem 9.10, we can see that $X \in \mathcal{E}_{\mathcal{R}}$. Hence, by using Corollary 7.8, we can infer that

$$\text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A))) = \text{int}_{\mathcal{R}^\Delta}(X) = X.$$

Therefore,

$$A \subseteq \text{int}_{\mathcal{R}^\Delta}(\text{cl}_{\mathcal{R}^\Delta}(\text{int}_{\mathcal{R}^\Delta}(A))),$$

and thus $A \in \mathcal{T}_{\mathcal{R}^\Delta}^\alpha$ trivially holds. Thus, we have proved that $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}^\Delta}^\alpha$. Hence, since $\emptyset \in \mathcal{T}_{\mathcal{R}^\Delta}^\alpha$ trivially holds, we can see that $\mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}^\Delta}^\alpha$ is also true. \square

Now, quite similarly, we can also prove the following

Theorem 35.2. *If \mathcal{R} is a non-partial, non-degenerated relator on X , then*

$$\mathcal{T}_{\mathcal{R}^\Delta}^\beta = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \text{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Remark 35.3. Thus, in contrast to Theorem 27.6, we usually have $\mathcal{T}_{\mathcal{R}^\Delta}^\alpha \not\subseteq \mathcal{T}_{\mathcal{R}^\Delta}^\beta$. This, in accordance with Remark 11.11, also shows that the relator \mathcal{R}^Δ is not, in general, reflexive.

By [132, Definition 31.1] relator \mathcal{R} on X to Y may be naturally called *hyperconnected* if $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$. That is, the identity function Δ_Y of Y is *fatness reversing* in the sense of [168, Definition 12.3].

Such relators have several remarkable properties. For instance, it can be easily shown that a relator \mathcal{R} on X to Y is hyperconnected if and only if $R(x) \cap S(y) \neq \emptyset$ for all $x, y \in X$ and $R, S \in \mathcal{R}$.

Thus, in particular, a hyperconnected relator is non-partial. Therefore, if \mathcal{R} is a non-degenerated, hyperconnected relator on X , then by Theorems 35.1 and 35.2 we already have $\mathcal{T}_{\mathcal{R}^\Delta}^\alpha \subseteq \mathcal{T}_{\mathcal{R}^\Delta}^\beta$.

Now, by using Corollaries 7.8 and 33.2, we can also prove the following

Theorem 35.4. *If \mathcal{R} is a non-partial, non-degenerated relator on X , then for any $A \subseteq X$ the following assertions are equivalent:*

- (1) $A \in \mathcal{T}_{\mathcal{R}^\Delta}^\gamma \setminus \{\emptyset\}$;
- (2) $A \in \mathcal{D}_{\mathcal{R}}$ and there exists $V \in \mathcal{E}_{\mathcal{R}}$ such that $A \subseteq V$.

Proof. If (1) holds, then $A \neq \emptyset$. Moreover, by Definition 27.1, there exists $V \subseteq X$ such that

$$V \in \mathcal{T}_{\mathcal{R}^\Delta}^s \quad \text{and} \quad A \subseteq V \subseteq \text{cl}_{\mathcal{R}^\Delta}(A).$$

Thus, in particular, $V \neq \emptyset$ and $\text{cl}_{\mathcal{R}^\Delta}(A) \neq \emptyset$ also hold. Hence, by using Corollaries 33.2 and 7.8, we can already infer that $V \in \mathcal{E}_{\mathcal{R}}$ and $A \in \mathcal{D}_{\mathcal{R}}$. Therefore, (2) also holds.

Conversely, if (2) holds, then by using Corollaries 33.2 and 7.8 we can see that

$$V \in \mathcal{T}_{\mathcal{R}^\Delta}^s \quad \text{and} \quad \text{cl}_{\mathcal{R}^\Delta}(A) = X.$$

Thus, in particular $V \subseteq \text{cl}_{\mathcal{R}^\Delta}(A)$ trivially holds. Hence, by Definition 27.1, we can see that $\mathcal{T}_{\mathcal{R}^\Delta}^\gamma$. Moreover, by Theorem 9.10, we can also note that $A \neq \emptyset$. Thus, (1) also holds. \square

Hence, analogously to Corollary 34.4, we can also derive

Corollary 35.5. *If \mathcal{R} is a non-partial, non-degenerated relator on X , then*

$$\mathcal{T}_{\mathcal{R}^\Delta}^\gamma = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \text{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Concerning the family $\mathcal{T}_{\mathcal{R}^\Delta}^\delta$, instead of an analogue of Theorem 35.4, we can only prove the following counterpart of Corollary 35.5.

Theorem 35.6. *If \mathcal{R} is a non-partial, non-degenerated relator on X , then*

$$\mathcal{T}_{\mathcal{R}^\Delta}^\delta = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \text{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}}.$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}^\Delta}^\delta$, then by Definition 27.1 there exists $V \subseteq X$ such that

$$V \in \mathcal{T}_{\mathcal{R}^\Delta}^p \quad \text{and} \quad V \subseteq A \subseteq \text{cl}_{\mathcal{R}^\Delta}(V).$$

Hence, if $A \neq \emptyset$, then we can infer that $\text{cl}_{\mathcal{R}^\Delta}(V) \neq \emptyset$. Thus, by Corollary 7.8, we necessarily have $V \in \mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{D}_{\mathcal{R}}$ is ascending, it is clear that $A \in \mathcal{D}_{\mathcal{R}}$ also holds. This already shows that $\mathcal{T}_{\mathcal{R}^\Delta}^\delta \subseteq \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\}$.

On the other hand, if $A \in \mathcal{D}_{\mathcal{R}}$, then by using Theorem 33.3 and Corollary 7.8, we can see that

$$A \in \mathcal{T}_{\mathcal{R}^\Delta}^p \quad \text{and} \quad \text{cl}_{\mathcal{R}^\Delta}(V) = X.$$

Hence, by taking $V = A$, we can see that

$$V \in \mathcal{T}_{\mathcal{R}^\Delta}^p \quad \text{and} \quad V \subseteq A \subseteq \text{cl}_{\mathcal{R}^\Delta}(V).$$

Thus, by Definition 27.1, we also have $A \in \mathcal{T}_{\mathcal{R}^\Delta}^\delta$. Hence, since $\emptyset \in \mathcal{T}_{\mathcal{R}^\Delta}^\delta$ is always true, we can already infer that $\mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \subseteq \mathcal{T}_{\mathcal{R}^\Delta}^\delta$. \square

36. A Further Illustrating Example and a Diagram

The following example will show that, for a non-reflexive relator \mathcal{R} , the six inclusions derivable from Theorems 19.6 and 27.5 need not be true.

Example 36.1. If $X = \{1, 2, 3\}$, and R_1 and R_2 are relations on X such that

$$\begin{aligned} R_1(1) &= \{1\}, & R_1(2) &= \{1\}, & R_1(3) &= \{2, 3\}; \\ R_2(1) &= \{1\}, & R_2(2) &= \{2, 3\}, & R_2(3) &= \{1, 2\}; \end{aligned}$$

then $\mathcal{R} = \{R_1, R_2\}$ is a non-partial relator on X such that:

- (1) $\mathcal{T}_{\mathcal{R}}^r = \{\emptyset, X\}$;
- (2) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^q \cup \{\{2, 3\}\}$;
- (3) $\mathcal{T}_{\mathcal{R}}^q = \mathcal{T}_{\mathcal{R}}^{ps} = \{\emptyset, \{1\}, \{1, 2\}, X\}$;
- (4) $\mathcal{T}_{\mathcal{R}}^s = \mathcal{T}_{\mathcal{R}}^p = \mathcal{T}_{\mathcal{R}}^a = \mathcal{T}_{\mathcal{R}}^b = \mathcal{T}_{\mathcal{R}}^\alpha = \mathcal{T}_{\mathcal{R}}^\beta = \mathcal{T}_{\mathcal{R}}^\gamma = \mathcal{T}_{\mathcal{R}}^\delta = \mathcal{T}_{\mathcal{R}}^q \cup \{\{1, 3\}\}$.

To check this, recall that for any $x \in X$ and $A \subseteq X$ we have

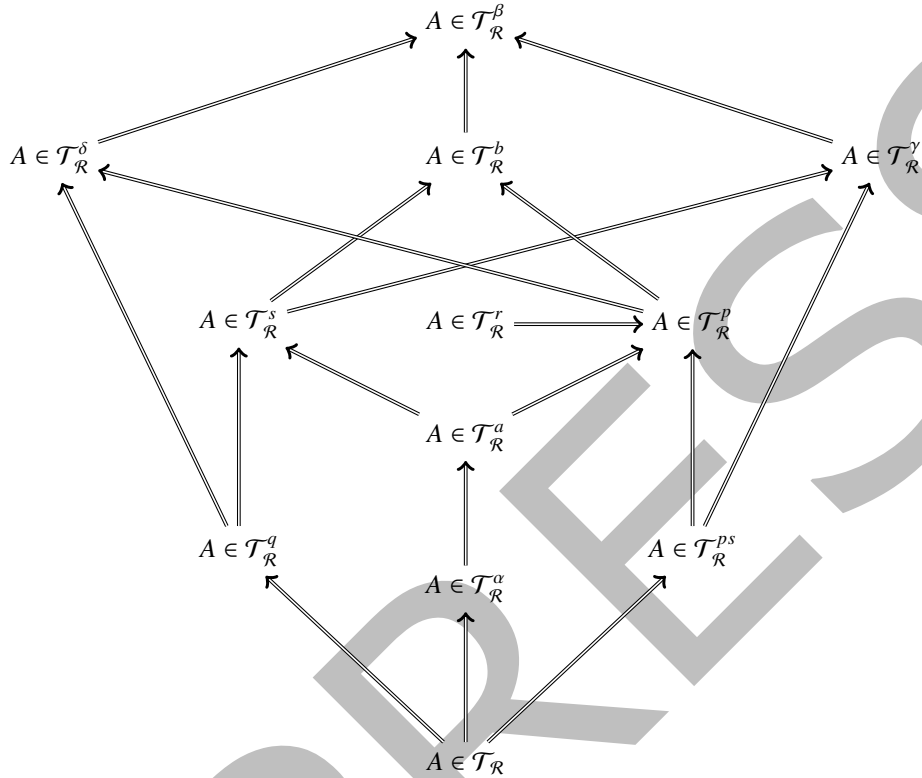
$$x \in A^\circ \iff \exists R \in \mathcal{R} : R(x) \subseteq A \quad (x \in A^- \iff \forall R \in \mathcal{R} : R(x) \cap A \neq \emptyset).$$

Therefore, concerning the relevant operations on subsets of X , we can state:

A	A°	A^-	$A^{\circ-}$	$A^{-\circ}$	$A^{\circ-\circ}$	$A^{-\circ-}$
\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$\{1\}$	$\{1, 2\}$	$\{1\}$	X	$\{1, 2\}$	X	X
$\{2\}$	\emptyset	$\{3\}$	\emptyset	\emptyset	\emptyset	\emptyset
$\{3\}$	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$\{1, 2\}$	X	X	X	X	X	X
$\{1, 3\}$	$\{1, 2\}$	X	X	X	X	X
$\{2, 3\}$	$\{2, 3\}$	$\{3\}$	$\{3\}$	\emptyset	\emptyset	\emptyset
X	X	X	X	X	X	X

By using Theorems 19.5, 19.6, 23.4, 27.3, 27.4, 27.5, 31.2 and 31.3, we can easily establish the following

Diagram 36.2. For a reflexive relator \mathcal{R} on X , the following implications hold :



Remark 36.3. Note that, by Theorems 19.5, 23.4 and 31.2, nine from the above implications do not require the relator \mathcal{R} to be reflexive.

37. Two Clarifying Examples to the Above Diagram

The following simple example will show only that eight implications in Diagram 1 are not reversible.

Example 37.1. If $X = \{1, 2, 3\}$ and R is a relation on X such that

$$R(1) = \{1, 2\} \quad \text{and} \quad R(2) = R(3) = X,$$

then $\mathcal{R} = \{R\}$ is a reflexive relator on X such that :

- (1) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^r = \mathcal{T}_{\mathcal{R}}^q = \{\emptyset, X\}$;
- (2) $\mathcal{T}_{\mathcal{R}}^s = \mathcal{T}_{\mathcal{R}}^a = \mathcal{T}_{\mathcal{R}}^p = \{\emptyset, \{1, 2\}, X\}$;
- (3) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^p = \mathcal{T}_{\mathcal{R}}^b = \mathcal{T}_{\mathcal{R}}^{\beta} = \mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{P}(X) \setminus \{\{3\}\}$.

To check this, note that now we have :

A	A°	A^-	$A^{\circ-}$	$A^{-\circ}$	$A^{\circ-\circ}$	$A^{-\circ-}$
\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$\{1\}$	\emptyset	X	\emptyset	X	\emptyset	X
$\{2\}$	\emptyset	X	\emptyset	X	\emptyset	X
$\{3\}$	\emptyset	$\{2, 3\}$	\emptyset	\emptyset	\emptyset	\emptyset
$\{1, 2\}$	$\{1\}$	X	X	X	X	X
$\{1, 3\}$	\emptyset	X	\emptyset	X	\emptyset	X
$\{2, 3\}$	\emptyset	X	\emptyset	X	\emptyset	X
X	X	X	X	X	X	X

The following, more difficult example will already show that sixteen implications in Diagram 1 are not reversible.

Example 37.2. If $X = \{1, 2, 3, 4\}$ and R_1 and R_2 are relations on X such that

$$\begin{aligned}
 R_1(1) = R_1(2) = \{1, 2, 3\}, & & R_1(3) = R_1(4) = \{1, 3, 4\}; \\
 R_2(1) = \{1, 2, 3\}, & & R_2(2) = \{1, 2\}, & & R_2(3) = R_2(4) = \{3, 4\};
 \end{aligned}$$

then $\mathcal{R} = \{R_1, R_2\}$ is a reflexive relator on X such that :

- (1) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^\alpha = \{\emptyset, \{3, 4\}, X\}$;
- (2) $\mathcal{T}_{\mathcal{R}}^r = \mathcal{T}_{\mathcal{R}} \cup \{\{2\}\}$;
- (3) $\mathcal{T}_{\mathcal{R}}^q = \mathcal{T}_{\mathcal{R}}^a = \mathcal{T}_{\mathcal{R}} \cup \{\{1, 3, 4\}\}$;
- (4) $\mathcal{T}_{\mathcal{R}}^s = \mathcal{T}_{\mathcal{R}}^g \cup \{\{1, 2\}\}$;
- (5) $\mathcal{T}_{\mathcal{R}}^p = \mathcal{P}(X) \setminus \{\{1\}, \{1, 2\}\}$;
- (6) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^p \setminus \{\{2\}\}$;
- (7) $\mathcal{T}_{\mathcal{R}}^b = \mathcal{T}_{\mathcal{R}}^\delta = \mathcal{P}(X) \setminus \{\{1\}\}$;
- (8) $\mathcal{T}_{\mathcal{R}}^\beta = \mathcal{T}_{\mathcal{R}}^\gamma = \mathcal{P}(X)$.

To check this, note that now we have :

A	A°	A^-	$A^{\circ-}$	$A^{-\circ}$	$A^{\circ-\circ}$	$A^{-\circ-}$
\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$\{1\}$	\emptyset	$\{1, 2\}$	\emptyset	$\{2\}$	\emptyset	$\{1, 2\}$
$\{2\}$	\emptyset	$\{1, 2\}$	\emptyset	$\{2\}$	\emptyset	$\{1, 2\}$
$\{3\}$	\emptyset	$\{1, 3, 4\}$	\emptyset	$\{3, 4\}$	\emptyset	$\{1, 3, 4\}$
$\{4\}$	\emptyset	$\{3, 4\}$	\emptyset	$\{3, 4\}$	\emptyset	$\{1, 3, 4\}$
$\{1, 2\}$	$\{2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{2\}$	$\{2\}$	$\{1, 2\}$
$\{1, 3\}$	\emptyset	X	\emptyset	X	\emptyset	X
$\{1, 4\}$	\emptyset	X	\emptyset	X	\emptyset	X
$\{2, 3\}$	\emptyset	X	\emptyset	X	\emptyset	X
$\{2, 4\}$	\emptyset	X	\emptyset	X	\emptyset	X
$\{3, 4\}$	$\{3, 4\}$	$\{1, 3, 4\}$	$\{1, 3, 4\}$	$\{3, 4\}$	$\{3, 4\}$	$\{1, 3, 4\}$
$\{1, 2, 3\}$	$\{1, 2\}$	X	$\{1, 2\}$	X	$\{2\}$	X
$\{1, 2, 4\}$	$\{2\}$	X	$\{1, 2\}$	X	$\{2\}$	X
$\{1, 3, 4\}$	$\{3, 4\}$	X	$\{1, 3, 4\}$	X	$\{3, 4\}$	X
$\{2, 3, 4\}$	$\{3, 4\}$	X	$\{1, 3, 4\}$	X	$\{3, 4\}$	X
X	X	X	X	X	X	X

Remark 37.3. Examples 37.1 and 37.2, together, already show that seventeen implications in Diagram 36.2 are not reversible.

However, unfortunately, they cannot be used to show that the remaining implication $A \in \mathcal{T}_{\mathcal{R}}^{\gamma} \implies A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ is also not reversible.

Note that, by Theorem 27.3, the above implication does not require the relator \mathcal{R} to be reflexive. Moreover, if \mathcal{R} is topological, then by Theorem 30.2 the reverse implication is also true.

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